Introduction to Topology

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ВВЕДЕНИЕ В ТОПОЛОГИЮ

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PREFACE

Topology is a subject that has only recently been introduced into the curriculum of mathematics departments. However, it does, in our opinion, play quite a considerable role with respect to university mathematics education as a whole. It is barely possible to design courses in mathematical analysis, differential equations, differential geometry, mechanics, and functional analysis that correspond to the modern state of these disciplines without involving topological concepts. It is therefore essential to acquaint students with topological methods of research as soon as they start their first university courses.

Having lectured on topology for some time to first year university students, we realised that there is a great need for a textbook that is comprehensible to students who have a minimal knowledge of mathematics (i.e., they are cognizant of general set theory, general algebra, the elements of linear algebra and mathematical analysis) and that will introduce a reader to the basic ideas underlying modern topology. At the same time, the book should contain a certain volume of topological concepts and methods.

This textbook is one of the many possible variants of a first course in topology and is written in accordance with both the authors' preferences and their experience as lecturers and researchers. It deals with those areas of topology that are most closely related to fundamental courses in general mathematics and applications. The material leaves a lecturer a free choice as to how he or she may want to design his or her own topology course and seminar classes.

We draw your attention to a number of devices we have used in this book in order to introduce general topology faster. We have therefore introduced constructive concepts, for example, those related to the notion of factor space, much earlier than the other notions of general topology. This makes it possible for students to study important examples of manifolds (two-dimensional surfaces, projective spaces, orbit spaces, etc.) as topological spaces. Later (Ch. IV) smooth structures are defined on them. The theory of two-dimensional surfaces is not confined to one place but is distributed amongst Ch. I, Ch. II and Ch. III as and when the basic ideas of general topology are developed. The notions of category and functor are introduced into homotopy theory quite early; likewise, the idea of the algebraization of topological problems. The functorial approach helps us expound homotopy and homology theories uniformly and complete the description of various homology theories with the Steenrod-Eilenberg axiomatics, making up, to some extent, for the absence of the proof of the invariance of simplicial homology theory in this textbook. Moreover, the homotopy computation technique (Ch. III) is reduced to the calculation of the fundamental groups of the circumference and closed sur-

faces. The equality $\pi_n (S^n; Z) = Z, n \geqslant 2$ is, however, given without proof and serves as a basis for the introduction of the degree of a mapping of spheres and the characteristic of a vector field (with the Brouwer and fundamental theorem of algebra being deduced); while in the homology group section (Ch. V), the technique is extended to exact sequences. In particular, the group $H_n (S^n; Z)$ is computed, and the Brouwer and Lefschetz fixed-point theorems are proved. In spite of having prepared everything for a development of the technique, we purposely leave the subject stand at that because of the special objectives of the textbook.

The concepts of smooth structure on a manifold and of tangent space (Ch. IV) have been elaborated as scrupulously as possible. The terminology has been revised, and the relations of the subject with mechanics, dynamic systems and the Morse theory emphasized. We believe that a number of homology theory variants (singular, simplicial, cellular) should be studied at as early a stage as possible when starting on the topic since the reader may encounter them in even the simplest applications. Accordingly, the variants are explained in Ch. V.

The exercises in the text of a section often replace a simple line of reasoning and are intended to invigorate the reader's thought. We indicate the end of the proof of a theorem with the sign ■, and if it is necessary to separate an example from the further text, the sign ♦ is used then.

Note that although this textbook is based on lectures delivered by Yu. G. Borisovich to students of the Mathematics Department of Voronezh University, they have been considerably revised by the authors. All the textual drawings are by T. N. Fomenko, while the cover and chapter title illustrations are by Prof. A. T. Fomenko (Moscow University) whom the authors would like to thank sincerely.

In conclusion, we would like to express our sincere gratitude to A. V. Chernavsky for his valuable advice and useful criticisms, to M. M. Postnikov both for our discussions on the methods of teaching topology to first-year university students and many other useful remarks, and, finally, may we thank A. S. Mishchenko for his useful advice.

We are also grateful to a group of young staff members and post-graduates at the department of algebra and topological methods in analysis of Voronezh University for their helpful discussions and remarks. Yu. E. Gliklich, V. G. Zvyagin, M. N. Krein and N. M. Meller have each checked through a number of chapters, and a considerable quantity of misprints were corrected due to their valuable assistance.

We have received many comments upon the Russian edition of this textbook and are thankful to those who have contributed them for many useful remarks they made. Many of their suggestions were taken into account when we prepared this revised edition.

The authors

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First Notions of Topology

The purpose of this chapter is to prepare the reader for the systematic study of topology as it is expounded in the subsequent chapters. Our purpose here is to review the problems, whose solution has led to the formation of topology as a mathematical discipline and to its development at present. We also discuss the beginnings of the notions of topological space and manifold.

1. WHAT IS TOPOLOGY?

Quant à moi, toutes les voies diverses où je m'étais engagé successivement me conduisaient à l'Analysis Sitûs*.

H. Poincaré

Topology as a science was, as it is generally believed, formed through the works of the great French mathematician Henri Poincaré at the end of the 19-th century. The beginnings of topology research may be dated to the work of G. Riemann in the middle of the 19-th century. In his investigations of the function theory, he developed new methods based on geometric representation. He is also known for having made an attempt to formulate the notion of many-dimensional manifold and to introduce higher orders of connectivity. These notions were developed by an Italian mathematician Betti (1871) but it was Poincaré who, by keeping in mind the requirements of the theories of function and differential equations, introduced a number of very important topological concepts, developed a profound theory and applied it to his research in these branches of mathematics and mechanics. His ideas and the problems which he suggested have had a considerable influence on the development of topology and its application up to the present day.

Poincaré defined topology (which was called then Analysis situs) as follows: 'L'Analysis Sitûs est la science qui nous fait connaître les propriétés qualitatives des figures géometriques non seulement dans 1' espace ordinaire, mais dans l'espace à plus de trois dimensions.

L'Analysis Sitûs à trois dimensions est pour nous une connaissance presque intuitive, L'Analysis Sitûs à plus de trois dimensions présente au contraire des difficultés enormes; il faut pour tenter de les surmonter être bien persuadé de l'extrême importance de cette science.

Si cette importance n'est pas comprise de tout le monde, c'est que tout le monde n'y a pas suffisamment réfléchi**' [62, 63].

To understand what is meant by the qualitative properties of geometric figures, imagine a sphere to be a rubber balloon that can be stretched and shrunk in any manner without being torn or 'gluing' any two distinct points together. Such transformations of a sphere are termed *homeomorphisms*, and the different replicas obtained as a result of homeomorphisms are said to be *homeomorphic* to one

* As for me, all the various journeys, on which one by one I found myself engaged, were leading me to Analysis situs (Position analysis).

** Analysis situs is a science which lets us learn the qualitative properties of geometric figures not only in the ordinary space, but also in the space of more than three dimensions. Analysis situs in three dimensions is almost intuitive knowledge for us. Analysis situs in more than three dimensions presents, on the contrary, enormous difficulties, and to attempt to surmount them, one should be persuaded of the extreme importance of this science. If this importance is not understood by everyone, it is because everyone has not sufficiently reflected upon it.

another. Thus, the qualitative properties of the sphere are those which it shares with all its homeomorphic replicas, or, in other words, those which are preserved under homeomorphisms.

It is evident that homeomorphisms and the qualitative properties of other figures may be discussed as well. It is also conventional to call the qualitative properties *topological properties.* In the above example, one of the topological properties of the sphere is obvious, i.e., its integrity (or connectedness). Its more subtle properties are revealed if an attempt is made to establish a homeomorphism of the sphere, say, with the ball. It is easy to conclude that such a homeomorphism is impossible. However, in order to prove that, it is necessary to show the various topological properties of spheres and balls. One of these is the 'contractibility' of the ball into one of its points by changing it 'smoothly', i.e., contracting it along its radii towards the centre, and the 'non-contractibility' of the sphere into any of its points. It is also wise to bear in mind the topological difference between a volleyball bladder and a bicycle tyre. These intuitive ideas need to be corroborated strictly.

Exercise 1°. Verify that the number of 'holes' in a geometric figure is its topological property; verify also that the annulus is not homeomorphic to the two-dimensional disc.

The research carried out by H. Poincaré is the starting point for one of the branches in topology, viz., combinatorial or algebraic topology. The method of investigation is to associate geometric figures using a rule that is common to all the figures with algebraic objects (e.g., groups, rings, etc.) so that certain relationships between the figures correspond to the algebraic relationships between the objects. Studying the properties of algebraic figures sheds light on the properties of geometric figures. The algebraic objects constructed by Poincaré are nothing but homology groups and the fundamental group.

The development of the method of algebraic topology inevitably led to concurrence with the ideas of set-theoretic topology (G. Cantor, between 1874-1895; F. Hausdorff, between 1900-1910). Even Poincaré himself set the task of generalizing the concept of geometric figure for spaces of more than three dimensions and of investigating their qualitative properties. This generalization induced the introduction of the notion of topological space, now a fundamental idea pervading all mathematics. It not only had an impact on the investigation of geometric figures in finite-dimensional spaces, but, due to the development of the theory of functions of a real variable and functional analysis, it also led to the construction of function spaces, which are, as a rule, infinite-dimensional.

'The first fairly general definitions of a topological space were given in the work by Fréchet, Riesz and Hausdorff. The complete definition of a topological space was given by the Polish mathematician K. Kuratowski and by P. S. Alexandrov' [42].

Topological spaces, their continuous mappings, and the study of general properties have made up one branch of topology known as 'general topology'.

The merger of the algebraic and set-theoretic schools in topology was accomplished by L. E. Brouwer in his work devoted to the notion of the dimension of a space (1908-1912). The unified approach was considerably developed by J. W. Alexander, S. Lefschetz, P. S. Alexandrov, P. S. Uryson, H. Hopf,

L. A. Lusternik, L. G. Schnirelman, M. Morse, A. N. Tihonov, L. S. Pontryagin, A. N. Kolmogorov, E. Čech, et al. Soviet mathematicians have made a profound and extensive contribution to the development of topology as a whole.

To describe precisely the results obtained (and even to pose problems) is impossible without being acquainted with the elements of general and algebraic topology. Here, we give merely some idea of the problems that have stimulated topological research.

If S^1 is a circumference on the Euclidean plane R^2 then the set $R^2 \setminus S^1$ decomposes into two mutually complementary open sets, viz., the interior A and the exterior B of S^1. The circumference S^1 serves as a separator between A and B. Can a simple continuous path be drawn from an arbitrary point $a \in A$ to an arbitrary point $b \in B$ so that it does not intersect the separator S^1? (A *simple continuous path* is a homeomorphic mapping of the line-segment $[0, 1]$ of the number line into the plane.) The answer is negative. In fact, if $\rho(x, y)$ is the Euclidean distance between points x, y of the plane R^2 and $\gamma(t)$ is such a path, $0 \leqslant t \leqslant 1$, $\gamma(0) = a$, $\gamma(1) = b$, then the function $f(t) = \rho(\gamma(t), 0)$, where 0 is the centre of the circumference, is continuous, and $f(0) < r$, $f(1) > r$, where r is the radius of the circumference S^1. By a property of continuous functions, $f(t)$ takes the value r at a point t_0. Therefore, $\gamma(t_0) \in S^1$.

Let us substitute a homeomorphic image Γ of the circumference S^1 (such a curve is said to be *simple closed*) for the circumference itself. A question arises as to whether $R^2 \setminus \Gamma$ can be partitioned into disjoint open sets so that the curve Γ remains the border of each of them. The answer is positive (the Jordan theorem), but the proof involves subtle topological concepts. And this time again the curve Γ is the separator between two open sets.

The problem gets still more complicated if a homeomorphic image of an n-dimensional sphere lying in the $(n + 1)$-dimensional Euclidean space is considered instead of a simple closed curve. The generalization of the Jordan theorem for this case was carried out by L. E. Brouwer in 1911-1913. A more extensive generalization of the result led to the creation of duality theorems (Alexander, Pontryagin, Alexandrov et al.) which shaped the development of algebraic topology for a long time thereafter.

Another important problem was the generalization of the concept of dimension. The dimension of a Euclidean space is well known as an algebraic concept, but is it also a topological concept? That is, will homeomorphic Euclidean spaces be of the same dimension? Lebesgue found in 1911 that the answer is positive.

As to geometric figures lying in Euclidean spaces, it is the notion of dimension that should have been formulated for them first. An idea concerning such a definition had been expressed by H. Poincaré himself. The dimension of the empty set is assumed to equal (-1). Now, by induction, if we know what is meant by dimensions less than or equal to $n - 1$, then the dimension n of a set signifies that it can be separated into parts that are as small as we please by a set of dimension $n - 1$ and cannot be partitioned by a set of dimension $n - 2$. These ideas were elaborated by Brouwer, Menger, Uryson, Alexandrov et al.

Another important direction in topology, which is closely related to applications, is fixed-point theory. We encounter, even in algebra and the elements of

analysis, the question whether or not there exist solutions of equations of the form

$$f(x) = 0, \tag{1}$$

where $f(x)$ is a polynomial or more complicated function. Equation (1) is equivalent to the equation

$$f(x) + x = x \tag{2}$$

or, when $F(x) = f(x) + x$, to the equation

$$F(x) = x. \tag{3}$$

The solutions of equation (3) are called the *fixed points of the mapping* F. If equation (1) is vector, i.e., if it is a system of equations in several unknows, then the equivalent equation (3) is also vector and, therefore, the fixed points lie in a many-dimensional Euclidean space.

An extremely important task is to find sufficiently general and effective tests that will indicate if fixed points exist. Brouwer obtained a remarkable result that had very extensive applications in modern research. It is surprisingly simple to formulate: any continuous mapping of a bounded, convex, closed set into itself has a fixed point. Convex sets may be considered both in the three-dimensional and many-dimensional Euclidean space. For example, a continuous mapping into itself of a closed (i.e., considered along with its boundary) disc in a plane or ball in a space necessarily has a fixed point.

Exercise 2°. Show that an analogue of the Brouwer theorem for an annulus does not hold true.

The Brouwer theorem was developed by H. Hopf, S. Lefschetz et al. It was also generalized for the mappings of function spaces (Kellog, Birkhoff, Schauder, Leray) which extended its applications. It should be noted that even H. Poincaré himself was interested in the existence theorems for fixed points when reducing certain problems in celestial mechanics to them.

We emphasize that the three problems described above do not make up the whole set of topological problems at all. Consider another example. Riemann introduced the notion of n-dimensional manifold, i.e., a space in which all the points possess n numerical coordinates defined at least on sufficiently small parts of the space. As a generalization of the notion of surface in the three-dimensional Euclidean space, the notion of manifold has embraced quite a number of geometric objects that arise from classical mechanics, differential equations and surface theory. Poincaré gave the final shape to the concept of manifold and developed elements of analysis for such spaces.

These concepts were elaborated in smooth manifold theory (G. de Rham, L. S. Pontryagin, H. Whitney et al.). Following the algebraic topology method, these spaces were associated with new algebraic objects, i.e., with 'the exterior differential form cohomology rings'. Smooth manifolds themselves were also 'organized' into a 'ring of interior homologies' (cobordism ring) (V. A. Rohlin). The accumulation of various algebraic objects in algebraic topology led later to the emergence and development of the so-called 'homological algebra'.

In the post-war period algebraic topology has been essentially restructured. By the beginning of the fifties many results of algebraic topology had been ac-

cumulated due to the research of such mathematicians as Hopf, Pontryagin, Whitney, Steenrod, Eilenberg, MacLane, Whitehead et al. It then became necessary to work out a unified approach to all the various data that have been obtained and to create new general methods. This restructuring of topology was generally influenced by the French topological school (Leray, Cartan, Serre et al.). In particular, this development led to the formally complete solution of the fundamental problem of homotopy theory (M. M. Postnikov), although the problems related to the determination of the homotopy type of concrete spaces are still a long way off solution.

The development of topology over the last 20 years has reached a high level in many directions. This process of developing has still not finished, although a number of the important problems that have faced topologists were solved. An active part in this has been taken by Soviet mathematicians. We cannot give even a short description of these directions, and so we shall only emphasize one important feature of modern topology, i.e., the very wide use of its methods in many of the other branches of modern mathematics such as those dealing with differential equations, functional analysis, classical mechanics, theoretical physics, general theory of relativity, mathematical economics, biology, etc. Topology has become a powerful instrument of mathematical research, and its language acquired a universal importance.

Details of the developments in topology may be found in *History of Soviet Mathematics* [42].

2. GENERALIZATION OF THE CONCEPTS OF SPACE AND FUNCTION

1. Metric Space. We have already mentioned that topology has worked out an essentially wider concept of space than that of Euclidean. We shall consider the notion of metric space (which is less general than that of topological space) as our first step both because of its greater simplicity and due to the wide use of this notion in modern mathematics.

In the Euclidean space R^3, the distance $\rho(x, y)$ is defined for each pair x, y of its points. The distance ρ possesses the following properties:

I. $\rho(x, y) \geqslant 0$ for any x, y.

II. $\rho(x, y) = 0$ if and only if $x = y$.

III. $\rho(x, y) = \rho(y, x)$.

IV. $\rho(x, y) \leqslant \rho(x, z) + \rho(z, y)$ for any $x, y, z \in R^3$ (the triangle inequality).

Thus, a distance is a real function of a pair of points (x, y) satisfying Properties I-IV. Functions like this can exist not only on R^3 but also on other kinds of sets.

Exercises.

1°. Let X be an arbitrary set. Put $\rho(x, y) = 0$ if x and y are coincident elements of X, and $\rho(x, y) = 1$ otherwise. Show that such a function ρ satisfies Properties I-IV.

2°. Let (ξ_1, ξ_2, ξ_3) be the coordinates of a point $x \in R^3$. Show that the function $\rho(x, y) = \max\limits_{1 \leqslant i \leqslant 3} |\xi_i - \eta_i|$, where (η_1, η_2, η_3) are the coordinates of a point $y \in R^3$, also satisfies Properties I-IV.

The functions ρ from Exercises 1 and 2 are naturally called the *distances between the elements* of the corresponding sets.

To introduce a general concept of distance, recall the definition of the product of two sets. If X and Y are two sets then their product $X \times Y$ is the set consisting of all ordered pairs (x, y), where $x \in X$, $y \in Y$. In particular, the product $X \times X$ is defined.

DEFINITION 1. A set X along with the mapping $\rho: X \times X \to R^1$ (into the number axis), associating each pair $(x, y) \in X \times X$ with a real number $\rho(x, y)$ and satisfying Properties I-IV, is called a *metric space* and denoted by (X, ρ).

The mapping ρ is called the *distance* or *metric on the space* X. The elements of X are usually called points.

Any set may be made into a metric space by endowing it with the metric described in Exercise 1°. Such a metric space is said to be *discrete*. However, this way of 'metrization' is not very effective.

EXAMPLE 1. Let $X \subset R^3$ be a subset of a Euclidean space. A distance in R^3 may simultaneously serve as a distance in X. The metric on X is obtained then by restricting the metric on R^3. ♦

If (X, ρ) is a metric space and $Y \subset X$ is a subset, then $(Y, \bar{\rho})$ is also a metric space, where $\bar{\rho}: Y \times Y \to R^1$ is a restriction of the mapping $\rho: X \times X \to R^1$.

The metric on Y is said to be *induced* by (to be hereditary from) the metric on X, and Y is said to be a *subspace of the metric space* X.

A number of examples of metric spaces naturally arise from problems in analysis.

EXAMPLE 2. Consider the set of all continuous functions on the line-segment $[0, 1]$. It is denoted usually by $C_{[0,1]}$. If $x(t)$, $y(t)$ are two continuous functions from $C_{[0,1]}$ then set

$$\rho(x, y) = \max_{t \in [0,1]} |x(t) - y(t)|. \tag{1}$$

Exercise 3°. Verify that function (1) is a metric. ♦

The set $C_{[0,1]}$ together with the metric described above is called the *space of continuous functions*; it plays an important role in analysis.

Exercises.

4°. Let A be an arbitrary set, and X the set of bounded real functions on A. If $f: A \to R^1$, $g: A \to R^1$ are arbitrary elements of X, then we put

$$\rho(f, g) = \sup_{t \in A} |f(t) - g(t)|.$$

Show that ρ is a metric on X.

5°. Let p be a prime number. If $n > 0$ is an integer and, when decomposed into prime factors, contains a power p^α, then we put $v_p(n) = \alpha$. Extend the function v_p from the set of positive integers to the set $Q \setminus 0$ of rational numbers without zero using the formula $v_p(\pm r/s) = v_p(r) - v_p(s)$. Put

$$\rho(x, y) = p^{-v_p(x-y)}, \quad x \neq y,$$
$$\rho(x, x) = 0$$

for arbitrary x, y from Q. Show that the function $\rho(x, y)$ is defined correctly and is a metric on Q (*p-adic distance*).

2. Convergent Sequences and Continuous Mappings. The notions that generalize the initial concepts of mathematical analysis can be introduced naturally for a metric space (X, ρ).

A mapping $n \to x_n$ of the set of natural numbers into a metric space (X, ρ) is called a *sequence of points* of this space and denoted by $\{x_n\}$. A sequence $\{x_n\}$ is said to *converge* to a point a (to have a limit a) if for any $\varepsilon > 0$, there is a natural number $n_0(\varepsilon)$ such that $\rho(x_n, a) < \varepsilon$ for all $n \geqslant n_0(\varepsilon)$.

This is often written thus:

$$x_n \xrightarrow{\rho} a, \text{ or just } x_n \to a.$$

Exercise 6°. Let $\{x_n = (\xi_1^n, \xi_2^n, \xi_3^n)\}$ be a sequence of points of the three-dimensional Euclidean space, and ρ the Euclidean metric. Prove that $x_n \xrightarrow{\rho} a$ if and only if $\xi_i^n \to \xi_i^0$ $(i = 1, 2, 3)$ as $n \to \infty$, where $a = (\xi_1^0, \xi_2^0, \xi_3^0)$.

By considering a sequence of continuous functions $x_n(t)$, $0 \leqslant t \leqslant 1$, to be a sequence in the metric space $C_{[0,\ 1]}$, we may speak of the convergence of this sequence to an element $x_0 = x_0(t)$: $x_n \xrightarrow{\rho} x_0$. Such a convergence is often said to be *uniform on the segment* $[0, 1]$.

Exercise 7°. Show that the sequence of functions $x_n(t) = n^2te^{-nt}$ on the segment $0 \leqslant t \leqslant 1$ converges to the zero function for any t but does not converge uniformly.

We now define the notion of continuous mapping of a metric space (X, ρ_1) into a metric space (Y, ρ_2).

DEFINITION 2. Let $f: X \to Y$ be a mapping of a set X into a set Y. If, for any point $x_0 \in X$ and any sequence $x_n \xrightarrow{\rho_1} x_0$ in X, the sequence of the images in Y converges to $f(x_0)$: $f(x_n) \xrightarrow{\rho_2} f(x_0)$, then the mapping f is called a *continuous mapping of the metric space* (X, ρ_1) into the metric space (Y, ρ_2).

This definition is evidently a generalization of the concept of continuous numerical function; it covers a great deal of mappings of geometric figures in Euclidean spaces.

If the property of continuity given by Definition 2 is considered at a certain point x_0, then a definition of a continuous mapping at the point x_0 is obtained.

Exercises.

8°. Let S^2 be a sphere in the Euclidean space R^3 with its centre at the origin. Putting $f(x) = -x$ (it is a point symmetry), prove that f is continuous.

9°. Give an example of a continuous mapping of a plane square into itself that has fixed points only on the boundary.

Obviously, an equivalent definition of a continuous mapping of metric spaces may also be given in terms of ε, δ.

A *mapping* $f: X \to Y$ *is continuous* if for any $x_0 \in X$ and for any $\varepsilon > 0$, there is $\delta = \delta(\varepsilon, x_0) > 0$ such that $\rho_2(f(x), f(x_0)) < \varepsilon$ as soon as $\rho_1(x, x_0) < \delta$.

If, in this definition, δ does not depend on the choice of the point x_0, then the mapping f is said to be *uniformly continuous*.

Exercise 10°. Let $f: R^1 \to R^1$ be a continuous function. Prove that the mapping $F: C_{[0,1]} \to C_{[0,1]}$, where $Fx(t) = f(x(t))$, is continuous.

Remember that a mapping of sets $f: X \to Y$ is said to be *surjective* if each element from Y is the image of a certain element from X; *injective* if different elements from X are mapped into different elements from Y; *bijective* if a mapping is both surjective and injective simultaneously.

We now have to define a homeomorphism of metric spaces.

DEFINITION 3. A mapping $f: X \to Y$ of metric spaces is called a *homeomorphism*, and the spaces X, Y *homeomorphic* if (1) f is bijective, (2) f is continuous, and (3) the inverse mapping f^{-1} is continuous.

This definition is a more precise way of expressing the idea of homeomorphic figures which we discussed intuitively in Sec. 1. Thus, the notion of the topological properties of figures also gains firmer ground: the *topological properties of metric spaces* are those which are preserved under homeomorphisms. Homeomorphic metric spaces are said to be *topologically equivalent*.

Exercises.

11°. Prove that (1) an annulus in R^2 is homeomorphic to a cylinder in R^3; (2) an annulus without boundary (the interior of an annulus) is homeomorphic to R^2 without one point, and to S^2 without two points.

12°. Show that the mapping of the half-interval $[0, 1)$ onto the circumference in the complex plane given by the function $z = e^{i2\pi t}$, $0 \leqslant t < 1$, is not a homeomorphism.

13°. Prove that (1) a closed ball and a closed cube in R^3 are homeomorphic; (2) the sphere S^2 with exclusion of the point N (this is the space $S^2 \setminus N$, where N is the north pole of the sphere) is homeomorphic to the plane R^2. (*Hint*: Use the stereographic projection.)

If a mapping $f: X \to Y$ is a homeomorphism onto its image $f(X)$ considered as a subspace in Y, then f is called an *embedding of the space X into Y*.

The following example of an embedding is often used: $X \subset Y$, $f(x) = x$.

3. FROM A METRIC TO TOPOLOGICAL SPACE

1. The 'Gluing' Method. We now discuss a concept of space more general than a metric space, viz., the concept of topological space, and give some initial ideas about such spaces. Firstly, we describe a way of constructing new spaces which immediately takes us outside the purview of metric spaces.

Let (X, ρ) be some metric space (in order to visualize one, think of X as a certain subset of the Euclidean space R^3). Let X be divided into disjoint subsets A_α:

$$X = \bigcup_\alpha A_\alpha; \quad A_\alpha \cap A_\beta = \varnothing \text{ if } \alpha \neq \beta.$$

If all the points from X that are in some A_α are called equivalent and 'glued' to one

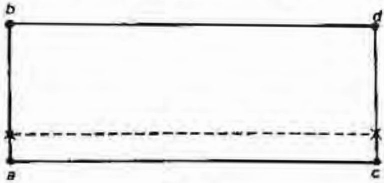


Fig. 1

point* a_α, then a new set $Y = \bigcup\limits_\alpha a_\alpha$ is obtained. This is called the *factor set* relative to a given equivalence. Note that Y is not a subset of X. Therefore, the metric ρ has, generally speaking, nothing to do with the 'space' Y.

A number of well-known surfaces in the Euclidean space can be obtained by gluing other surfaces together. Consider some of them. Let X be a rectangle (Fig. 1). If those points on *ab* and *cd* which lie on a common horizontal, are 'pasted' together, then a factor set that can be identified with a cylinder is obtained (Fig. 2). If the points on the sides *ab* and *dc* that are symmetric with respect to the centre O of the rectangle are 'glued' together then a Möbius strip is obtained (Fig. 3).

A Möbius strip can be made of a sheet of paper by pasting together the opposite sides in an appropriate manner. This can be used to demonstrate visually a number of the properties of the Möbius strip.

The Möbius strip has many remarkable properties: it has one edge (the closed line *adbca*) and, in contrast to a cylinder, it has one side because it can be painted one colour with a continuous movement of the brush without passing over the edge (these properties are easy to see on a paper model). The Möbius strip is a non-orientable surface. Remember that a surface is said to be orientable if any sufficiently small circle on the surface with a fixed direction of the journey along its

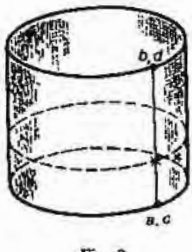


Fig. 2

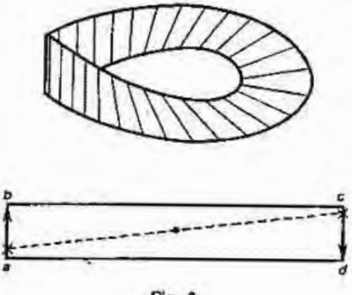


Fig. 3

* Strictly speaking, this means that each set A_α of equivalent points from X is considered as one element of the new set.

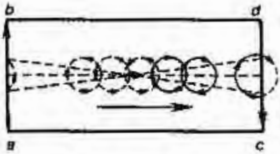


Fig. 4

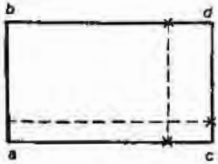


Fig. 5

Fig. 6

boundary, during any 'smooth' shifting across its surface, preserves the original direction of the journey along the boundary (the circle is assumed not to intersect the edge of the surface); otherwise, the surface is termed non-orientable. The non-orientability of the Möbius strip is clear from Fig. 4.

If the sides ab and cd of the sheet $abcd$ are pasted together so that each point of ab meets that point of cd which lies on the same horizontal, and at the same time the sides bd and ac are pasted together so that the points on common verticals meet, then we obtain a surface called a *torus** (Fig. 5).

If, however, ab and cd, as well as bd and ac, are glued together so that the points which are symmetric with respect to the centre O meet (Fig. 6), then the factor set cannot be represented as a figure in the three-dimensional Euclidean space. More exactly, such an attempt to paste equivalent points together would lead to a surface that would pierce itself without selfintersecting. We could only place this surface in R^3 by tearing it apart in a convenient manner, but this would violate our tacit principle of the 'continuity' of gluing (i.e., the points that are near to equivalent points remain near points after gluing again). The obtained factor set is called the *projective plane* and denoted by RP^2.

Note that the rectangle $abdc$ is homeomorphic to a disc with the boundary $abdc$, and the projective plane can also be described as a disc (Fig. 7) whose diametrically opposite boundary points are glued together or, finally, as a hemisphere whose diametrically opposite boundary points are glued together into one point (Fig. 8).

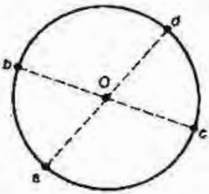


Fig. 7

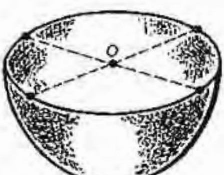


Fig. 8

* If not stated otherwise, by torus the surface is meant throughout the book (*tr.*).

Thus, forming the factor sets in the first three cases leads to figures in the Euclidean space R^3 again, and gives a new object in the last case.

Exercises.

1°. Verify that cylinders, tori, and spheres are orientable surfaces, whereas the projective plane is non-orientable.

2°. Obtain, by an appropriate gluing operation (factorization), a circumference from a line-segment, a sphere from a disc, a circumference from R^1, and a torus from R^2.

2. **On the Notion of Topological Space.** We can see now how the idea of a topological space comes into play. We mentioned above that a factor set cannot always be placed in a metric space in a natural manner and therefore a metric deduced for the set. One of the functions of a metric is to characterize how near two points are, and in the definition of a continuous mapping, a metric plays this role (cf. Sec. 2). To geometrize the notion of nearness, consider a ball

$$D_r(x_0) = \{x \in X: \rho(x, x_0) < r\}, \quad r > 0,$$

with centre at the point x_0 and radius r. A point x is ε-near to the point x_0 if $x \in D_\varepsilon(x_0)$.

It is easy to verify that the continuity of a mapping $f: X \to Y$ of two metric spaces can be described in an equivalent manner thus: let $x_0 \in X$ be an arbitrary (fixed) point and $y_0 = f(x_0)$ an element of Y; then for any ball $D_\varepsilon(y_0)$, there is a ball $D_\delta(x_0)$ such that $f(D_\delta(x_0)) \subset D_\varepsilon(y_0)$.

The continuity of a mapping may be now said to signify the preservation of the nearness of points. The concept of nearness allows us to formulate exactly that of neighbourhood of a point: a part Ω of a metric space is a neighbourhood of a point x_0 in Ω if each point, which is sufficiently near to x_0, belongs to Ω. Thus, neighbourhood structures arise in metric spaces.

'Nevertheless, the spaces so defined have a great many properties which can be stated without reference to the 'distance' which gave rise to them. For example, every subset which contains a neighbourhood of x_0 is again a neighbourhood of x_0, and the intersection of two neighbourhoods of x_0 is a neighbourhood of x_0. These properties and others have a multitude of consequences which can be deduced without any further recourse to the 'distance' which originally enabled us to define neighbourhoods. We obtain statements in which there is no mention of magnitude or distance' [18, p. 12].

If, in a set X, a distance is not introduced, then the nearness does not have an exact meaning and the above definition of a neighbourhood is inappropriate. However, the inverse process proves effective, i.e., each element $x_0 \in X$ is associated with a certain collection of subsets $\{\Omega(x_0)\}$ of the set X so as to fulfil the main properties (axioms) of neighbourhoods. This collection is then called a system of neighbourhoods and the elements from the neighbourhood $\Omega(x_0)$ are said to be Ω-near to x_0. The set X is then said to be endowed with a *topological structure*, or a topology and called a *topological space* whilst the elements of X are called points.

'Once topological structures have been defined, it is easy to make precise the idea of continuity. Intuitively, a function is continuous at a point if its value varies

as little as we please whenever the argument remains sufficiently near the point in question. Thus continuity will have an exact meaning whenever the space of the argument and the space of values of the function are topological spaces' [18, p. 13].

Thus, replacing the balls in the definition of a continuous mapping by neighbourhoods, we obtain the notion of continuous mapping, and then the definition of a homeomorphism of topological spaces. Homeomorphic topological spaces are called *topologically equivalent*.

EXAMPLE. Let C be the complex plane. The extended plane of a complex variable $\bar{C} = C \cup \infty$ is a topological space: the spherical neighbourhoods of points $z \in C$ and the neighbourhoods of the point ∞ of the form

$$D_r(\infty) = \{z \in C: |z| > r\} \cup \infty,$$

and also subsets containing them, form a topological structure on $\bar{C}$.

Exercise 3°. Prove that the extended complex plane is homeomorphic to the sphere S^2, the north pole N being the image of the point ∞, and the south pole the image of the point 0.

Hint: Use the stereographic projection $S^2 \setminus N$ onto the equatorial plane C, viz.,

$$u = \frac{x + iy}{1 - Z}, \quad (x, y, z) \in S^2 \setminus N.$$

In the case of factor sets, a topological structure naturally emerges from a topological structure in a metric space by gluing neighbourhoods together. Thus, a factor set becomes a topological space (*factor space*).

3. **Gluing Two-Dimensional Surfaces Together.** Let us study in more detail the factor spaces that are obtained by gluing together plane figures. Consider a polygon Π in the plane R^2 and induce a metric from R^2 on it. The spherical neighbourhoods of a point $x \in \Pi$ obviously consist of the intersections with Π of open discs having centres at x. Thus sufficiently small spherical neighbourhoods of the point x are open discs if x does not lie on the boundary of the polygon, and are sectors of an open disc (together with the boundary radii) if x lies on the boundary (Fig. 9).

Let there be two polygons Π and Π'. Mark one side of each, a and a', respectively. Π and Π' can be glued along these sides, thus producing the homeomorphism $\alpha: a \to a'$. We declare the image and the inverse image equivalent. The topology of the factor space $(\Pi \cup \Pi')/R$ respective to this equivalence consists of open discs for

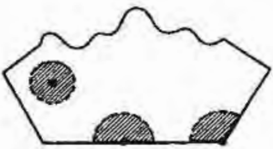


Fig. 9

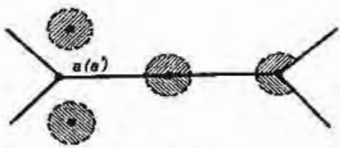


Fig. 10

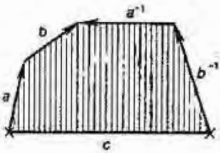


Fig. 11

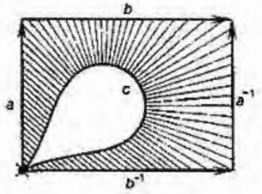


Fig. 12

Fig. 13

interior points ($x \in \Pi$, $x' \in \Pi'$), the sectors glued together for equivalent points $x \in a$, $x' \in a'$, and the sets containing the mentioned neighbourhoods. Fig. 10 illustrates the case when identifying is done by joining the polygons along the equal sides a and a'.

Similarly, two sides of the same polygon can be glued together (see the examples in Item 1).

Exercises.

4°. Describe the topologies of a cylinder, torus, Möbius strip, and projective plane.
5°. Verify that the examples of factor spaces (in Item 1) are homeomorphic to their realizations in the Euclidean space R^3. Verify that different models (Figs. 6, 7 and 8) of the projective plane are homeomorphic.

Consider now the gluing of surfaces. Let us paste together the sides of a pentagon as shown in Fig. 11. The arrow-heads denote the gluing rule for corresponding sides (the beginning of one oriented line-segment is glued to the beginning of another, and the end of the former to the end of the latter). The designations a, a^{-1} remind us of the necessity, when identifying equivalent points, to traverse along the side a clockwise on the boundary of the polygon, and counterclockwise along the side a^{-1} (in other words, the sides with opposite orientation are glued). The gluing scheme is described by the formula $aba^{-1}b^{-1}c$. It is easy to see that this factor space may also be obtained in another topologically equivalent way (Fig. 12): here, the factor space is represented by a torus with a cut-out along the curve c (Fig. 13, the dotted line denotes where aa^{-1} and bb^{-1} were glued). A torus with a hole is termed a handle.

Consider the gluing of two adjacent sides of a triangle. If the orientations are opposite, i.e., the gluing scheme is $aa^{-1}c$ (Fig. 14), then the factor space is topologically equivalent to a sphere with a hole (Fig. 15).

Consider the gluing of two adjacent sides with the same orientation, i.e., according to the scheme aac (Fig. 16). We represent this triangle as two right triangles glued together along the common height d (Fig. 17) with the indicated orientation. Now we change the order in which they are glued: firstly, identify the hypotenuses a, and then the legs d (Fig. 18). A Möbius strip (cf. Fig. 3) is obtained, the final factor space being homeomorphic to the original (Fig. 16).

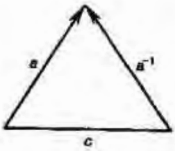


Fig. 14

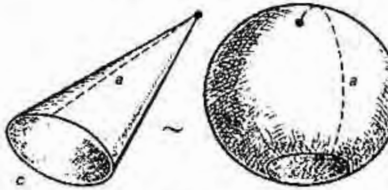


Fig. 15

Now, if a disc is cut out of the sphere S^2, then either a handle or a Möbius strip can be glued along the available edge c. The latter can be represented as the circumference S^1 (the boundary of the circle which has been cut out). In the first case, a torus is obtained (Fig. 19) (verify the topological equivalence of the figures in the drawing), in the second the projective plane RP^2. Let us verify this.

The projective plane (see Fig. 8) is topologically equivalent to the factor space drawn in Fig. 20. In fact what remains to show is that the upper 'cap' is a Möbius strip with the edge c. Representing it as an annulus where diametrically opposite points of the inner circumference are identified, we can topologically transform it to a Möbius strip (Fig. 21).

The further constructions can be continued in two ways: (1) by cutting p discs out of the sphere and gluing p handles to them; (2) by cutting out q discs and gluing q Möbius strips to them. Thus, two series of surfaces may be obtained:

$$M_0, M_1, M_2, \ldots, M_p, \ldots, N_1, N_2, \ldots, N_q, \ldots \tag{1}$$

(M_0 and N_0 are obviously the sphere S^2).

We shall discuss the properties of these surfaces. First of all, it is easy to see that they are obtained from a finite number of convex polygons by gluing together their sides and subsequent topological transformations. Such spaces are said to be *finitely triangulable*, and partitioning of the space into 'curvilinear' polygons is called a *triangulation*. The surfaces M_p, N_q are connected, meaning that they consist of a whole 'piece' and do not split into two mutually exclusive sets of polygons. This follows from the fact that any two vertices of the triangulation polygons are joined

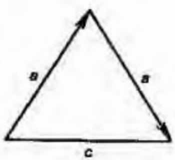


Fig. 16

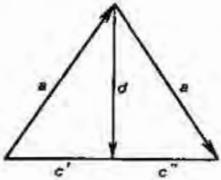


Fig. 17

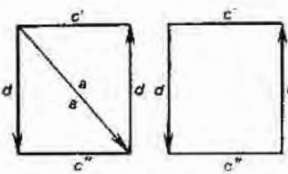


Fig. 18

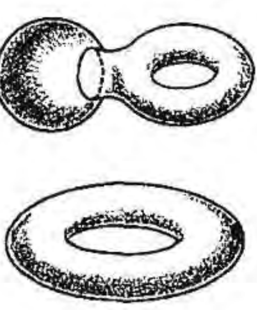


Fig. 19

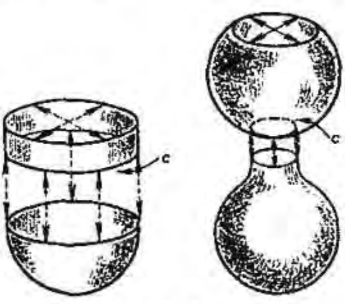


Fig. 20

by a continuous path made up of their sides. The surfaces under consideration do not have a boundary, since each boundary side of the polygon has been glued to one (and only one) side. Hence, each point of such a surface has a neighbourhood homeomorphic to an open disc; these spaces are called *two-dimensional manifolds*.

Finitely-triangulable connected two-dimensional manifolds are said to be *closed surfaces*. If we did not glue together all the pairs of the sides of the polygons, and left some sides free, we would obtain a *nonclosed surface* (or a *surface with boundary*). A point on the boundary would have a neighbourhood homeomorphic to a semicircle. An example is the sphere S^2 with several holes.

Note also that the surfaces M_p are orientable, and can be placed into R^3 as two-sided surfaces without self-intersections. By contrast, the surfaces N_q are non-orientable (they are called one-sided like the Möbius strip) and cannot be embedded in R^3 without self-intersections (but can be in R^4!).

It is shown in Ch. II that any closed surface is homeomorphic to an M_p- or N_q-type surface (the numbers p, q are called the *genus of the surface*). The surfaces M_p and N_q, $q \geqslant 1$, are never homeomorphic, since the orientability of a surface is a topological property. Two different M_p- and $M_{p'}$-surfaces (or N_q- and $N_{q'}$-surfaces) cannot be homeomorphic either (see the next item). Thus, the list in (1) is a complete topological classification of closed surfaces. If p handles and $q \geqslant 1$

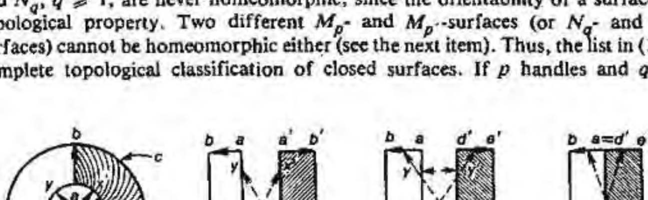


Fig. 21

Möbius strips (having made $p + q$ holes) are glued to a sphere then the obtained surface is topologically equivalent to a sphere to which $2p + q$ Möbius strips are glued.

Exercises.

6°. Glue a cylinder to the boundaries of a sphere with two holes. Prove that the obtained surface is homeomorphic to a sphere with a handle glued to it, i.e., a torus.
7°. Show that an annulus and a Möbius strip can be obtained from a disc by gluing the boundary of the latter to two sides of a rectangle.
8°. Prove the equivalence of the following definitions of RP^2 to those given above: (i) diametrically opposite pairs of points are identified in S^2; (ii) the edge of a Möbius strip is shrunk into one point; (iii) the edge of a Möbius strip is glued to a disc by a certain homeomorphism of the boundary circumferences.
9°. Define RP^1 by identifying diametrically opposite points of the circumference S^1. Show that (i) RP^1 is homeomorphic to the circumference S^1; (ii) $RP^1 \subset RP^2$; (iii) there is a neighbourhood of RP^1 in RP^2 which is homeomorphic to a Möbius strip.
10°. Prove the equivalence of the following definitions of the Klein bottle: (i) two Möbius strips glued along their boundaries; (ii) an annulus with the boundary circumferences glued together and with their circumnavigation directions reversed; (iii) an annulus to each of whose boundaries a Möbius strip is glued.

A topological space which is homeomorphic to a convex polygon is called a *topological polygon*. Accordingly, we define the images of the vertices (resp. sides) to be the *vertices* (resp. edges) of the topological polygon. Without loss of generality, a triangulation of a surface may be assumed to consist of topological polygons which are edgewise adjacent to each other (to carry this out, the convex polygons whose sides are identified to obtain the surface should be divided a priori into sufficiently small polygons such as triangles). Hereafter, only such triangulations are considered.

For any triangulated surface Π, we define the number $\chi(\Pi) = e - k + f$, where e is the number of vertices, k the number of edges and f the number of the triangulation polygons, which is known as the *Euler characteristic of the surface* Π. It possesses the remarkable property that it does not depend on a triangulation, i.e., is a topological invariant of the surface.

Exercise 11°. Verify that the Euler characteristic of the sphere S^2 equals 2, that of a torus zero, that of a disc unity, that of a handle minus one, and that of a Möbius strip zero.

It is easy to prove the topological invariance of $\chi(S^2)$ using the Jordan theorem* which states that any simple closed curve, i.e., a curve which is homeomorphic to the circumference, splits the sphere or the plane into two disjoint regions, their boundary coinciding with the curve.

Thus, consider a triangulation of S^2. It can be achieved gradually by fixing a vertex (•) and drawing one edge after another; we draw the first edge from the vertex (•) to a new vertex, and then take care that each subsequent edge should

* The proof of the Jordan theorem is quite long, and we do not give it here.

begin at the vertex of an edge already drawn. We count the number of the obtained vertices e, edges k, and regions f that are bounded by a simple closed curve made up of the edges at each step. At first, we put $e = 1, k = 0, f = 1$ (when we have the vertex (*) and its complementary region). It is easily seen that the number $e - k + f$ remains unaltered after adding each new edge. In fact, if an edge comes to a new vertex then no new regions emerge and the numbers e and k are both increased by 1. If the new edge joins two of the original vertices then it will close a certain edge path and a new region will emerge (by the Jordan theorem). Thus, k and f will both be increased by unity, and e will remain unaltered. Having drawn the last edge, we shall completely restore the triangulation, and then $e - k + f = \chi(S^2)$. Originally, $e - k + f = 2$, therefore, $\chi(S^2) = 2$.

If Π_1, Π_2 are two surfaces with boundaries l_1, l_2 which are homeomorphic to S^1, then they can be glued boundariwise in accordance with the homeomorphism $\alpha: l_1 \to l_2$. Let $\Pi_1 \cup_\alpha \Pi_2$ denote a factor space. We shall prove the formula

$$\chi(\Pi_1 \cup_\alpha \Pi_2) = \chi(\Pi_1) + \chi(\Pi_2). \tag{2}$$

Let us triangulate Π_1 and Π_2 so as to obtain homeomorphic triangulations (the triangulation of S^1 being made up of l vertices and the same number of edges) on the boundaries l_1, l_2. After gluing, the number of vertices is equal to $e_1 + e_2 - l$, that of the edges to $k_1 + k_2 - l$, and that of the polygons to $f_1 + f_2$. Formula (2) follows from the equality

$$(e_1 + e_2 - l) - (k_1 + k_2 - l) + (f_1 + f_2) = (e_1 - k_1 + f_1) + (e_2 - k_2 + f_2).$$

Formula (2) is sometimes convenient for calculating the Euler characteristic.

Let ${}_pS^2$ be a sphere with p holes. If p discs are glued back, then we obtain S^2. Formula (2) yields the equality $\chi(S^2) = \chi({}_pS^2) + p$, whence $\chi({}_pS^2) = 2 - p$.

The surface M_p is obtained by gluing ${}_pS^2$ to p handles whose Euler characteristic equals -1. From (2), we obtain that $\chi(M_p) = 2 - 2p$. Similarly, $\chi(N_q) = 2 - q$, since the Euler characteristic of a Möbius strip equals zero. Since $\chi(M_{p_1}) = \chi(M_{p_2})$ only if $p_1 = p_2$, and $\chi(N_{q_1}) = \chi(N_{q_2})$ only if $q_1 = q_2$, the surfaces M_{p_1}, M_{p_2} cannot be homeomorphic when $p_1 \neq p_2$ due to the topological invariance of the Euler characteristic, nor can the surfaces N_{q_1}, N_{q_2} be homeomorphic when $q_1 \neq q_2$.

Other interesting applications of the Euler characteristic can be found in the theory of convex polyhedra. The surface of a convex polyhedron can be imagined to have been glued together from a finite number of convex polygons (its faces) respective to the identity mappings of the edges glued. We immediately obtain the Euler formula for a convex polyhedron:

$$\alpha_0 - \alpha_1 + \alpha_2 = 2,$$

where α_0 is the number of vertices, α_1 the number of edges, and α_2 the number of faces of the polyhedron. In fact, the left-hand side is the Euler characteristic of the surface of the polyhedron which is obviously homeomorphic to S^2.

If m faces meet at each vertex, and each face is a convex n-gon, then the polyhedron is said to be of type $\{n, m\}$. If the n-gons are regular then the polyhedron

is said to be *regular*. If the type $\{n, m\}$ is known, then $\alpha_0, \alpha_1, \alpha_2$ can be calculated. In fact, m edges meet in each vertex, therefore $\alpha_0 m = 2\alpha_1$; since there are n edges in each face, $\alpha_2 n = 2\alpha_1$ (each edge joins two vertices and two faces). Thus,

$$\frac{\alpha_0}{m^{-1}} = \frac{\alpha_1}{2^{-1}} = \frac{\alpha_2}{n^{-1}} = \frac{\alpha_0 - \alpha_1 + \alpha_2}{m^{-1} - 2^{-1} + n^{-1}}$$

$$= \frac{2}{m^{-1} - 2^{-1} + n^{-1}} = \frac{4mn}{2n + 2m - mn},$$

whence the values of α_0, α_1, and α_2 can be calculated. The natural requirement for α_0, α_1, and α_2 to be positive leads to the inequality for positive integers n, m:

$$2n + 2m - nm > 0 \sim (n - 2)(m - 2) < 4.$$

It is easy to conclude that there are five solutions all in all:

$$\{3,3\}, \{4,3\}, \{3,4\}, \{5,3\}, \{3,5\}. \tag{3}$$

Five kinds of regular polyhedra are known from elementary geometry, viz., tetrahedron, cube, octahedron, dodecahedron, and icosahedron whose types are precisely what is given in (3).

Thus, a complete list of the $\{n, m\}$-type polyhedrons has been given.

4. THE NOTION OF RIEMANN SURFACE

One of the ways leading to the basic topological concepts is to study algebraic functions and their integrals; it was discovered by Riemann as early as the middle of the last century.

Consider the algebraic equation

$$a_0(z)w^n + a_1(z)w^{n-1} + \ldots + a_n(z) = 0, \quad a_0(z) \neq 0, \tag{1}$$

with complex coefficients that are polynomials of a complex variable z; its roots are functions $w = w(z)$ in z, and analytic under certain conditions. For example, if all the roots of equation (1) are different at a point z_0, then there exist n functions $w_i(z)$, $i = 1, \ldots, n$ in the neighbourhood of the point z_0 that depend on z analytically.

An analytic function $w = w(z)$ satisfying equation (1) is called an *algebraic function*. Equation (1) determines several branches $w_i(z)$ of algebraic functions whose number will vary, generally speaking, and which change into one another as z varies. Consequently, mathematicians speak of a many-valued algebraic function $w(z)$ determined by equation (1) and of its branches $w_i(z)$. Riemann proposed that the z-plane C be replaced by a surface on which the function $w(z)$ will be one-valued, while its branches $w_i(z)$ will be the values of $w(z)$ on separate parts of the surface (such surfaces are called *Riemann surfaces*).

It is not complicated to construct such a surface. Consider the extended complex plane $\bar{C} = C \cup \infty$ (the z-sphere) and the Cartesian product $\bar{C} \times \bar{C}$ consisting of ordered pairs (z, w). Neighbourhoods in $\bar{C} \times \bar{C}$ are defined, naturally, as the Carte-

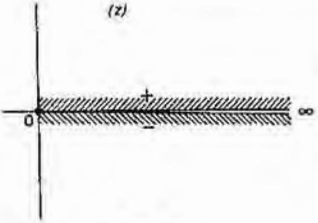


Fig. 22

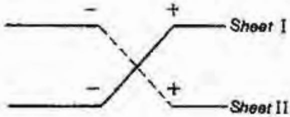


Fig. 23

sian products of neighbourhoods (and as all the sets that contain them). Then algebraic equation (1) determines a graph in $\bar{C} \times \bar{C}$ consisting of those pairs (z, w) which satisfy equation (1). This is the Riemann surface Π of the many-valued algebraic function $w(z)$; in fact, the projection $\Pi \to \bar{C}$ given by the rule

$$(z, w) \to w, \tag{2}$$

determines a one-valued function on a Riemann surface which takes the values of all the branches of the many-valued function. An interesting question arises about the structure of the surface Π and about the distribution of the branches of the function w on it.

The simplest many-valued algebraic function is related to the equation of the second degree

$$w^2 + a_1(z)w + a_2(z) = 0. \tag{3}$$

The change of variables $v = 2w + a_1$ reduces this equation to a simpler form

$$v^2 - p(z) = 0, \tag{3'}$$

where $p(z)$ is a polynomial.

In the simplest case, $p(z) = z$. Then equation (3') determines the two-valued algebraic function $w = \sqrt{z}$. If $z = re^{i\varphi}$ then its two values $w_1 = \sqrt{r}e^{i\varphi/2}$, $w_2 = -\sqrt{r}e^{i\varphi/2}$, $r > 0$, have opposite signs and change into each other when the point z moves along a closed path around the point $z = 0$. To prevent the reduction of the branch w_1 into the branch w_2, cut the z-sphere along the positive real half-axis (Fig. 22). This cut joins the points 0 and ∞. Two edges abut on the cut, viz., the upper (+) and the lower (−). Consider the union (disjoint) of sheet I and sheet II (replicas) of the z-sphere cut. Call sheet I the carrier of the branch w_1, and sheet II the carrier of the branch w_2. On the two-sheeted surface I ∪ II, the function w is one-valued. To detect the effect of the reduction of the branch w_1 into the branch w_2, we glue the (−)-edge of sheet I to the (+)-edge of sheet II, and the (+)-edge of sheet I to the (−)-edge of sheet II. We obtain a factor space Π'_1 which is the two-sheeted Riemann surface of the function $w = \sqrt{z}$. Although not lying in R^3 (sheets I and II pierce each other, see the gluing scheme, Fig. 23), it gives us a good visual demonstration of the relationship between the branches w_1 and w_2.

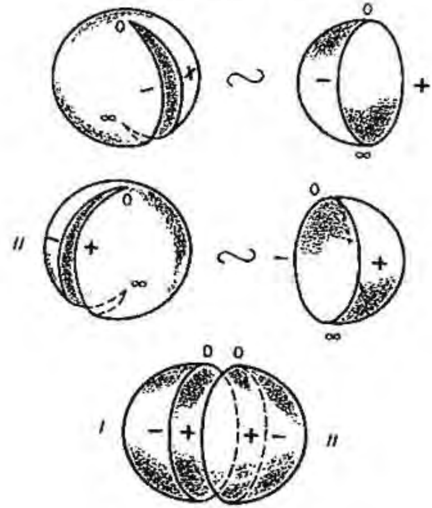


Fig. 24

But, for the algebraic equation $w^2 - z = 0$, the graph Π_1 in $\bar{C} \times \bar{C}$, on which the function w is also one-valued, is determined.

We now show that Π_1 and Π_1' are homeomorphic to $\bar{C}$, i.e., to the two-dimensional sphere S^2. In fact, mapping (2)

$$w = w(t), \text{ where } t = (z, w) \in \Pi_1,$$

is easily seen to be a homeomorphism, as well as the mapping $w: \Pi_1' \to \bar{C}$ which is induced by the many-valued function $w = \sqrt{z}$. Therefore, the Riemann surface of the many-valued algebraic function $w = \sqrt{z}$ is topologically equivalent to the two-dimensional sphere S^2. This, by the way, can be seen in Fig. 24 when sheets I and II were glued after transforming them topologically and a priori into hemispheres by 'moving' their edges further away from each other.

Let us specify another projection $\Pi_1 \to \bar{C}$ by the formula $z(t) = z$ and identify $\bar{C}$ with S^2. We then have two diagrams.

$$\begin{tikzcd} \Pi_1 \arrow[rr, "w=w(t)"] \arrow[dr, "z=z(t)"'] & & S^2(w\text{-sphere}) \arrow[dl, "z=w^2"] \\ & \begin{matrix} S^2 \\ (z\text{-sphere}) \end{matrix} & \end{tikzcd} \qquad \begin{tikzcd} \Pi_1 \arrow[dr, "z=z(t)"'] & & S^2(w\text{-sphere}) \arrow[ll, "t=(w^2, w)"'] \arrow[dl, "z=w^2"] \\ & \begin{matrix} S^2 \\ (z\text{-sphere}) \end{matrix} & \end{tikzcd}$$

These diagrams are commutative, i.e., the superposition of two mappings (in the direction of the arrow-heads) equals the third mapping (the remaining arrow). Both the horizontal mappings in the diagram are homeomorphisms inverse to each other.

The mapping $S^2 \xrightarrow{z = w^2} S^2$ is termed a *two-sheeted (ramified)* covering of the sphere S^2 with the branch points $z = 0$ and $z = \infty$ (verify that circumnavigation of the point $z = \infty$ also leads to a change of the branch).

This covering gives rise to substitutions rationalizing the integrands in familiar integrals of the form $\int R(z, \sqrt{\ })dz$, where $R(z, \sqrt{\ })$ is a rational function in z and $\sqrt{\ }$.

Consider the simplest integral

$$\int_{z_0}^{z} R(z, \sqrt{z})ds, \tag{5}$$

regarded as a curvilinear integral in the z-plane C along a certain path $z = z(s)$, $0 \leqslant s \leqslant 1$, joining the points z_0 and z, where $\sqrt{z}$ is one of the branches of the many-valued algebraic function given by the equation $w^2 - z = 0$. The same integral can also be considered as the curvilinear integral

$$\int_{\bar{\gamma}} R(t)dt$$

along the path $\bar{\gamma}$: $t = (z(s), \sqrt{z(s)})$ in the space $C \times C$, joining the points $(z_0, \sqrt{z_0})$, $(z, \sqrt{z})$, of the function $R(t) = R(z, w)$. Let $\bar{\gamma}$ lie in a Riemann surface Π_1. However, it is more convenient to consider its image in a surface which is homeomorphic to Π_1, i.e., the w-plane $\bar{C}$, the image being the path $\tilde{\gamma}$: $w = w(s)$, where $w(s) = \sqrt{z(s)}$. The relation between the z-plane $\bar{C}$ and the w-plane $\bar{C}$ is specified by the transformation $z = w^2$ and makes it possible to transform integral (5) by the change of variables $z = w^2$ into the integral along the curve $\tilde{\gamma}$ in the w-plane $\bar{C}$:

$$\int_{z_0}^{z} R(z, \sqrt{z})dz = \int_{w_0}^{w} R(w^2, w)2wdw.$$

The last integral is of a rational function. Thus, we rationalized the integrand by sending (using a two-sheeted covering of the z-sphere S^2) the path of integration onto the Riemann surface of the many-valued algebraic function determined by the equation $w^2 - z = 0$.

Now, let $p(z) = a_0z^2 + a_1z + a_2$, where $a_0, a_1, a_2 \in C$, $a_1^2 - 4a_0a_2 \neq 0$, $a_0 \neq 0$. Denoting the roots of the polynomial $p(z)$ by r_1, r_2, where $r_1 \neq r_2$, we obtain the algebraic function

$$w = \sqrt{a_0(z - r_1)(z - r_2)}. \tag{6}$$

Obviously, it is also two-valued. An investigation similar to that above shows that one branch is reduced to the other during the circumnavigation of both the point r_1 and the point r_2, while the circumnavigation of both these points (along a closed path surrounding the points r_1 and r_2) and the point ∞ does not alter the number of a branch. Therefore, the Riemann surface Π_2' of this function is obtained from the two replicas of the z-sphere which are cut along the line-segment $\overline{r_1r_2}$, the edges of sheets I and II being glued together as they were in the first example. Evidently, the space Π_2' is still topologically equivalent to the sphere. We have again a two-sheeted

covering of the sphere S^2 with two branch points $z = r_1$, $z = r_2$.

The graph Π_2 of the algebraic equation

$$w^2 - a_0(z - r_1)(z - r_2) = 0 \tag{6'}$$

is also homeomorphic to the sphere S^2. In fact, the homeomorphism $\tau = \dfrac{z - r_1}{z - r_2}$ of the z-sphere into the τ-sphere reduces r_1 into $\tau_1 = 0$, and r_2 into $\tau_2 = \infty$.

The change of variables

$$v = \frac{w}{\sqrt{a_0}(r_1 - r_2)}(1 - \tau),\ \tau = \frac{z - r_1}{z - r_2}$$

transforms, as can be easily seen, algebraic equation (6') into the algebraic equation $v^2 - \tau = 0$, the corresponding mapping Φ: $\bar{C} \times \bar{C} \to \bar{C} \times \bar{C}$, $(z, w) \to (\tau, v)$, reduces Π_2 into Π_1 and is a homeomorphism, and the homeomorphism Π_1 onto S^2 is given by projection (2): $v = v(t)$, where $t = (\tau, v)$.

Thus, we have the commutative diagram

$$\begin{tikzcd}
\Pi_2 \arrow[r, "\Phi"] \arrow[d] & \Pi_1 \arrow[r, "v=v(t)"] \arrow[d] & S^2\,(v\text{-sphere}) \arrow[dl, "\tau=v^2"] \\
\underset{(z\text{-sphere})}{S^2} \arrow[r, "\tau=\frac{z-r_1}{z-r_2}"] & \underset{(\tau\text{-sphere})}{S^2} &
\end{tikzcd} \tag{7}$$

If the integral

$$\int_{z_0}^{z} R(z, \sqrt{a_0(z - r_1)(z - r_2)})dz = \int_{z_0}^{z} R(z, w)dz$$

is given on Π_2, then horizontal mappings of diagram (7) enable us to transform it into the integral

$$\int_{v_0}^{v} \bar{R}(v^2, v)dv,$$

on the v-sphere S^2, where $\bar{R}$ is a rational function. This accounts for the rationalization of the integrand by the formal Euler substitution

$$v = \sqrt{\tau} = \sqrt{\frac{z - r_1}{z - r_2}}.$$

We will come to an essentially new result if we consider a polynomial $p(z)$ of the third degree. Thus, consider an algebraic function of the form

$$w = \sqrt{a_0(z - r_1)(z - r_2)(z - r_3)} \tag{8}$$

where r_1, r_2, r_3 are pairwise different. The function w possesses two branches, but now they are 'joined' in a more complicated way. The circumnavigation of one

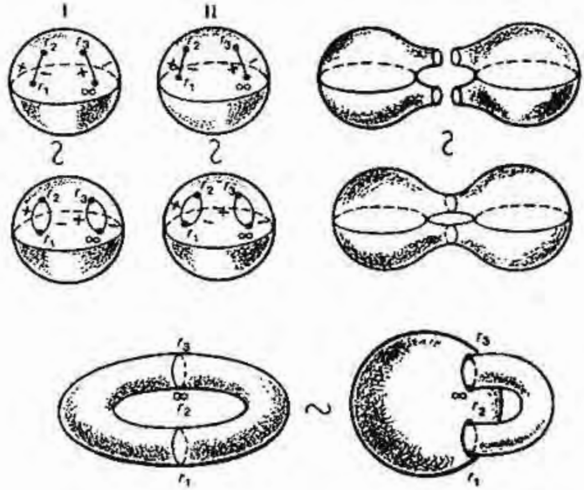


Fig. 25

point r_i results in a change of a branch of the function w, while that of any two points preserves the branch, and that of all the three points, just like the circumnavigation of the point ∞, alters the branch. To 'ban' these transformations, it suffices to make the cuts $\overline{r_1 r_2}$ and $\overline{r_3 \infty}$ on the z-sphere. Then each branch of the function w is one-valued on such a sheet with the cuts. For one branch to be transformed into the other in the required way, we glue replicas I and II along the cuts $\overline{r_1 r_2}$ and $\overline{r_3 \infty}$, respectively, the edges being glued as before. The topological space Π_3' obtained is evidently the Riemann surface of function (8). An essential difference between the surface Π_3' and the surface Π_2' is that Π_3' is topologically equivalent to a sphere with a handle (Fig. 25, where the cuts are first expanded into 'holes' from which tubes are pulled and glued edgewise together in the required way). The natural mapping $\Pi_3' \to \bar{C}$ is a two-sheeted covering map of S^2 with the branch points r_1, r_2, r_3, ∞.

For the function $w = \sqrt{a_0(z - r_1)(z - r_2)(z - r_3)(z - r_4)}$, where r_1, r_2, r_3, r_4 are pairwise different, we have the Riemann surface Π_4' which is homeomorphic to Π_3'. This follows from the one-valued branches being separated by the two cuts $\overline{r_1 r_2}$ and $\overline{r_3 r_4}$ and the point r_4 acting as the point ∞ of the previous example (the latter not being a branch point).

Note that integrating rational functions on the surfaces Π_3, Π_4 leads to elliptic integral theory.

It is not complicated either to investigate the case of an algebraic function

$$w = \sqrt{a_0(z - r_1) \dots (z - r_n)}, \tag{9}$$

where r_i are pairwise different. Here, $n/2$ cuts are made, i.e., $\overline{r_1r_2}, \dots, \overline{r_{n-1}r_n}$, if n is even, and $(n + 1)/2$ cuts $\overline{r_1r_2}, \dots, \overline{r_{n-2}r_{n-1}}, \overline{r_n\infty}$, if n is odd. Having taken two replicas of the z-sphere with such cuts, we glue them along the corresponding cuts. The constructions are similar to those indicated in Fig. 25 and will produce a sphere with $\left(\frac{n}{2} - 1\right) = \frac{n-2}{2}$ or $\frac{n+1}{2} - 1 = \frac{n-1}{2}$ handles. This is the Riemann surface of function (9). The number of handles p (the genus of a surface) is related to the number V of the branch points of the Riemann surface by the equality $V = 2(p + 1)$.

Thus, the many-valued algebraic function, which is determined by equation (3), possesses a Riemann surface that is topologically equivalent to a sphere with handles. This statement is valid for any many-valued algebraic function.

Exercise 1°. Construct the Riemann surface of the algebraic function $w^n - z = 0$, where $n > 2$ and is an integer, and verify that it is n-sheeted and topologically equivalent to the sphere.

The investigation of non-algebraic analytic functions in the z-plane also leads to Riemann surfaces on which the analytic functions are one-valued.

Exercise 2°. Consider the logarithmic function determined by the equation $e^w - z = 0$ and construct its Riemann surface.

5. SOMETHING ABOUT KNOTS

Intuitively, the notion of knot seems uncomplicated. The simplest examples are the 'prime' knot (Fig. 26) and the 'figure-of-eight' knot (Fig. 27) which can be easily represented with a rope. Any attempt to transform a 'prime' knot into a 'figure-of-eight' knot without passing the ends of the rope through a loop will fail. Thus, such an experiment shows that these knots are different, which brings up the subject of classifying knots mathematically.

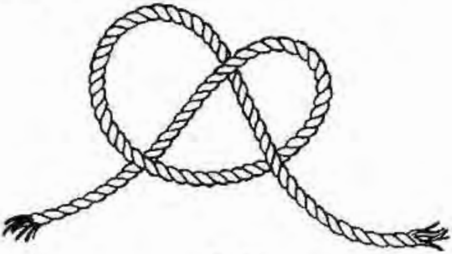


Fig. 26

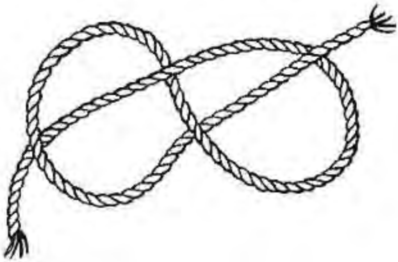


Fig. 27

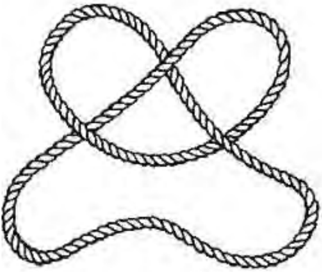


Fig. 28

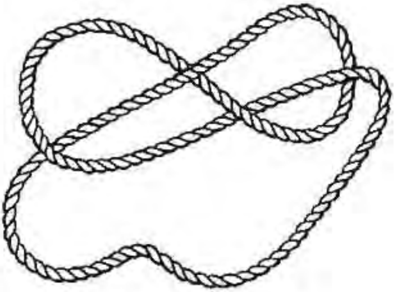


Fig. 29

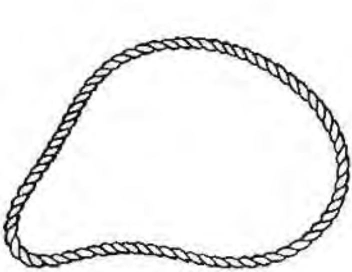


Fig. 30

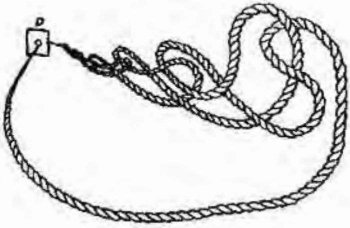


Fig. 31

We shall be able to ban passing the ends through a loop if we identify (i.e., glue together) the rope-ends (Fig. 28 and Fig. 29). Then the following definition becomes natural.

DEFINITION 1. A knot is a homeomorphic image of the circumference S^1 in R^3.

EXAMPLES (a) the trivial knot (Fig. 30); (b) a 'prime' knot, 'cloverleaf' knot or 'trefoil' knot (see Fig. 28), (c) a 'figure-of-eight' or a 'fourfold' knot (see Fig. 29). ♦

Note that by definition all knots are homeomorphic. Therefore, it is required to classify the embeddings (homeomorphisms) by which the circumference can be embedded in R^3.

DEFINITION 2. *Knots K_1 and K_2 are said to be equivalent* if there exists a homeomorphism of R^3 onto itself mapping K_1 onto K_2.

A more exact classification of knots is based on the notion of isotopy of the space R^3. A continuous mapping H: $[0, 1] \times R^3 - R^3$ is called an *isotopy* if for each $t \in [0, 1]$, the mapping H homeomorphically maps R^3 onto itself, whereas, when $t = 0$, it is the identity mapping. Thus, an isotopy is a family of homeomorphisms of the space R^3 which depend on a parameter t and which change continuously as t increases, beginning with the identity when $t = 0$.

DEFINITION 3. Knots K_1 and K_2 are of the same *isotopy type* if there exists an isotopy $H(t, x)$ of the space R^3, $t \in [0, 1]$, $x \in R^3$ such that $H(1, K_1) = K_2$.

Exercise 1°. Show that the belonging to an isotopy class is an equivalence relation.

There are examples of knots which are equivalent in the sense of Definition 2 but are of different isotopy types. Thus a 'trefoil' knot and the mirror image of a 'trefoil' knot, i.e., the knot which is symmetric to the 'trefoil' knot with respect to some plane in R^3, are not of the same isotopy type (the proof of this requires the development of a special technique). However, a 'figure-of-eight' knot and its mirror image are of the same isotopy type.

The basic properties of knots are easily studied for knots that are comparatively simply tied.

DEFINITION 4. A *polygonal knot* is a knot which is the union of a finite number of rectilinear segments.

DEFINITION 5. A knot which is equivalent to a polygonal one is said to be *tame*. A knot which is not equivalent to a polygonal one is said to be *wild*.

EXAMPLES. The trivial, 'trefoil', and 'figure-of-eight' knots are tame. An example of a wild knot is given in Fig. 31. The number of loops in this knot increases indefinitely whereas their size decreases indefinitely while approaching the point p. It is interesting that if the number of loops were finite, then the knot would be equivalent to the trivial one. ♦

Knot classification is closely related to properties of spaces which are complementary to knots. For example, if some of the topological invariants of the complements of knots K_1 and K_2 are different then K_1 is not equivalent to K_2 (and not isotopic). A useful topological invariant is the fundamental group of the knot complement (the knot group) (see Ch. III). Note also that the set of all knot equivalence classes (or isotopy equivalence classes) may be endowed with an algebraic structure. The idea of such a structure may be given in the following manner: call the *composition (product)* $K_1 * K_2$ *of two knots* K_1, K_2 the operation of tying them one after the other. The order in which they are tied is immaterial; more exactly, the knot $K_1 * K_2$ is equivalent to the knot $K_2 * K_1$. The composition of the knot equivalence classes so defined is commutative and associative. The equivalence class of the trivial knot serves as the identity element. However, an attempt to solve the equation $K * X = 1$ (i.e., to untie K by tying the knot X) will fail except when $K = 1$. Therefore, the knot equivalence classes only form a semigroup (and do not form a group).

FURTHER READING

The first topological notions can be learned from many sources. A systematic introduction to a number of basic concepts (including two-dimensional and three-dimensional manifolds) is carried out by Efremovich in *Encyclopedia of Elementary Mathematics*, V. 5 *Geometry. Basic Concepts of Topology* [29] (pp. 476-556). *Visual Topology* [17] by Boltyansky and Efremovich may be quite useful for the beginner. It explains the ideas, basic notions and findings of topology in a popular manner. Also *First Concepts of Topology* [22] by Chinn and Steenrod should be noted.

The problems of gluing together two-dimensional surfaces are also covered in popular books: *What is Mathematics*? [23] (Ch. V) by Courant and Robbins, *Anschauche Geometrie*, [39] (Ch. VI) by Hilbert and Cohn-Vossen, *New Mathematical Diversions from Scientific American* [34], and *The Unexpected Hanging and Other Mathematical Diversions* [35], etc. by Gardner. Gardner explains how Möbius strips are used for grinding. When we introduced the Euler characteristic, we also used some techniques from *An Account of the Basic Ideas of Topology* by Boltyansky and Efremovich [16], and some from Coxeter [24].

The classification of two-dimensional surfaces is covered very thoroughly in *Algebraic Topology: An Introduction* [52] (Ch. 1 and Ch. 2) by Massey and also in the *Lehrbuch der Topologie* [71] (Ch. II) by Seifert and Threlfall.

Metric spaces and their mappings are dealt with in *Introduction to Set Theory and General Topology* [3] (Ch. 4) by Alexandrov and in the text-books on functional analysis, [46] (Ch. II), [49] (Ch. I) and [26].

An elementary approach to the idea of a topological space may be found in the two books mentioned at the beginning of this survey. Note also in this connection 'Introduction' and 'Historical Note' to Ch. I from the book by Bourbaki *Topologie générale* [18] which we quoted in Sec. 3.

The first concepts of complex variable function theory that we referred to in Sec. 4 may be found, for example, in [32] and development of the notions introduced in Sec. 4 can be found in [75] and [49].

Knot theory, whose basic ideas we discussed in Sec. 5, is covered by *Introduction to Knot Theory* [25] by Crowell and Fox.

Note a little available textbook by Chernavsky and Matveyev [20] which was published by Voronezh State University and which contains elements of general topology and homotopy theory, the topology of surfaces, elements of manifold theory as well as short surveys of basic definitions.*

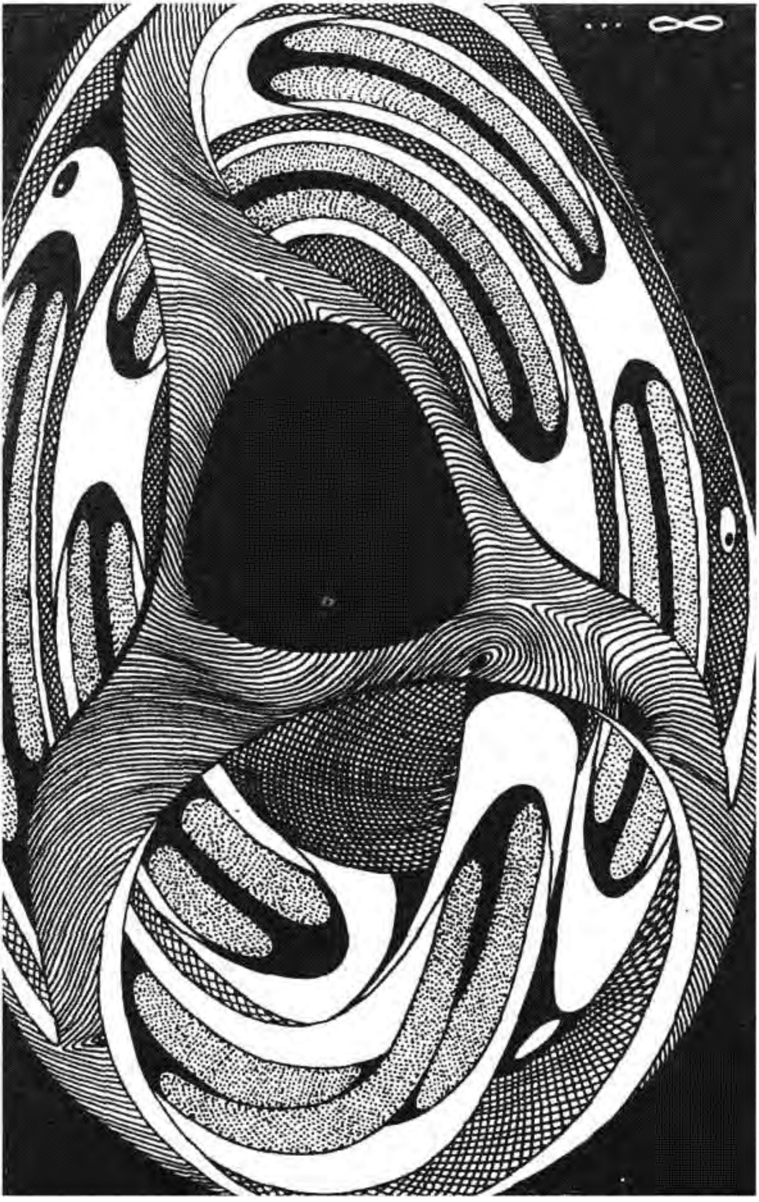
We indicate, in conclusion, that the third volume of *History of Soviet Mathematics* [42] contains a historical survey of the development of topology in this country.

* A number of the problems given in Ch. I are from this textbook.

II

General Topology

As we have mentioned above, the notion of metric space is insufficient for the development of a number of important mathematical problems. In the twentieth century, a more general concept of space has arisen and developed in mathematics, the concept of topological space. By now, this notion has become universally accepted since the 'structure' of a topological space, a concept quite broad and profound, usually precedes the introduction of other structures. The language of topological space theory has become generally accepted in all the branches of mathematics which are related to the notion of space. This chapter is devoted to the theory of topological spaces and their continuous mappings.



1. TOPOLOGICAL SPACES AND CONTINUOUS MAPPINGS

1. The Definition of a Topological Space. Let there be a collection of subsets $\tau = \{U\}$ in a set X of an arbitrary nature so that it possesses the following properties:

(i) $\varnothing, X \in \tau$;

(ii) the union of any collection of sets from τ belongs to τ;

(iii) the intersection of any finite number of sets from τ belongs to τ.

Such a collection of subsets τ is called a *topology in* X; the set X is called a *topological space* and denoted by (X, τ), and the subsets from the collection τ are said to be *open* (in the space X).

EXAMPLES.

1. X is the number line. A topology is given by the following collection of subsets: the empty set $\varnothing$, all possible intervals and their unions $U = \bigcup_\alpha (a_\alpha, b_\alpha)$.

2. $X = R^2$. Call a set open if together with each of its points, it contains a sufficiently small open circle centred at the point, and the empty set. It is easy to verify that the family of all open sets in R^2 forms a topology.

3. X is an arbitrary set. Put $\tau_0 = \{\varnothing, X\}$. It is a topology (verify!). Thus, (X, τ_0) is a topological space.

4. X is an arbitrary set, $\tau_1 = \{$all possible subsets of $X\}$. It is also a topology (verify!). ◆

The topology τ_1 is said to be *maximal* or *discrete*, and the topology τ_0 is called *minimal* or *trivial*. Thus, different topologies, e.g., the trivial or discrete one, may be defined on the same set.

The dual notion of *closed set* is closely related to the notion of open set: it is a set whose complement is open. Thus, if $U \in \tau$ then $X \setminus U$ is closed, and, conversely, if F is closed then $X \setminus F$ is open.

Exercise 1°. Verify that the following sets are closed: a line-segment $[a, b]$ in R^1; and a closed disc in R^2.

As a result of the duality of set-theoretic operations, the collection of all closed sets of a space X satisfies the following properties:

(i) the sets $X, \varnothing$ are closed;

(ii) the intersection of any collection of closed sets is closed;

(iii) the union of any finite number of closed sets is closed.

Various topologies on the same set form a partially ordered set.

DEFINITION 1. A topology τ is said to be *weaker* (*coarser*) than a topology τ' $(\tau < \tau')$ if it follows from $U \in \tau$ that $U \in \tau'$; i.e., if all sets from τ belong to τ'. The topology τ' is then said to be *stronger* (*finer*) than the topology τ.

Note that for any topology τ, we have $\tau_0 < \tau < \tau_1$. It is clear that there also exist incomparable topologies. Topologies τ' and τ'' are incomparable if each of them contains only some of the sets belonging to the other.

We will now consider how to construct a topology. First, an important definition.

DEFINITION 2. A collection $B = \{V\}$ of open sets is called a *base for a topology* τ, if for any open set U and for any point $x \in U$, there exists a set $V \in B$ such that $x \in V$ and $V \subset U$.

Therefore, any non-empty open set in X can be represented as the union of open sets from the base. This property characterizes a base. In particular, X equals the union of all the sets from V (any collection of sets with such a property is called a *covering* of the space). Conversely, if a set X is represented as the union $X = \bigcup_\alpha V_\alpha$ then under what conditions can a topology on X be constructed so that the family $B = \{V_\alpha\}$ is a base for the topology?

THEOREM 1 (A CRITERION OF A BASE). *Let* $X = \bigcup_\alpha V_\alpha$. *A covering* $B = \{V_\alpha\}$ *is a base for a certain topology if and only if for any* V_α, *any* V_β *from* B *and any* $x \in V_\alpha \cap V_\beta$, *there exists* $V_\gamma \in B$ *such that* $x \in V_\gamma \subset V_\alpha \cap V_\beta$.

PROOF. If $B = \{V_\alpha\}$ is a base for a topology then $V_\alpha \cap V_\beta$ is an open set, and, by the definition of a base, for any $x \in V_\alpha \cap V_\beta$, there exists $V_\gamma : x \in V_\gamma \subset V_\alpha \cap V_\beta$. Conversely, if $B = \{V_\alpha\}$ satisfies the condition of the theorem then the sets $U = \cup V_\alpha$ (all possible unions) and the empty set $\varnothing$ form, as can easily be verified, a topology on X for which $B = \{V_\alpha\}$ is a base. ■

Note that we have also indicated in the proof a way of constructing a topology if a family B satisfying the condition of the theorem is given.

But can a topology on the set X be constructed for an arbitrary covering $\{S_\alpha\}$? The following theorem answers this question.

THEOREM 2. *A covering* $\{S_\alpha\}$ *naturally generates a topology on* X, *viz., the collection of sets* $\{V = \bigcap_{\alpha \in K} S_\alpha\}$, *where* K *is an arbitrary finite subset from* $\{\alpha\}$, *is a base for the topology.*

PROOF. Verify that the collection $\{V\}$ satisfies the criterion of a base. In fact, put $V_\gamma = V_\alpha \cap V_\beta$ for $V_\alpha \cap V_\beta$. Obviously, $V_\gamma \in \{V\}$, and, therefore, the criterion of a base is fulfilled. ■

Thus, the covering $\{S_\alpha\}$ of the set X determines a topology on X whose open sets are all the possible unions $\cup(\bigcap_{\alpha \in K} S_\alpha)$ and the empty set.

DEFINITION 3. The family $\{S_\alpha\}$ is called a *subbase* for the topology which it generates.

EXAMPLES.

5. Let $X = R^1$. Sets of the form $S_\alpha = \{x : x < \alpha\}$, $\alpha \in R^1$, and $S_\beta = \{x : x > \beta\}$, $\beta \in R^1$, form a subbase for the topology of the number line R^1.

6. Let $X = R^n$ be an n-dimensional vector space. A base is a collection of sets $B = \{V_{a,b}\}$ in R^n, where $V_{a,b} = \{x \in R^n : a_i < \xi_i < b_i, i = 1, \ldots, n\}$, ξ_i is the i-th coordinate of the vector $x = (\xi_1, \xi_2, \ldots, \xi_n)$; $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are arbitrary vectors in R^n, $a_i < b_i$.

Sets like $V_{a,b}$ are called *open parallelepipeds* in R^n. ♦

Exercise 2°. Prove that the set of parallelepipeds described in Example 6 forms a base for the topology on R^n.

It is natural, for a topological space, to select a base with the least possible number of elements. For example, sets $V = (t_1, t_2)$ in R^1, where t_1, t_2 are rational, form a base consisting of a countable set of elements.

Similarly, there is a countable base for R^n consisting of parallelepipeds with rational vertices of the form

$$V_{r_1 r_2} = \{x : r_1^i < \xi_i < r_2^i, \quad i = 1, \ldots, n\},$$

where r_1, r_2 are rational vectors in R^n.

2. Neighbourhoods.

Let (X, τ) be a topological space, and $x \in X$ an arbitrary point.

DEFINITION 4. A *neighbourhood of a point* $x \in X$ is any subset $\Omega(x) \subset X$ satisfying the conditions: (i) $x \in \Omega(x)$, (ii) there exists $U \in \tau$ such that $x \in U \subset \Omega(x)$.

We may consider the collection of all neighbourhoods of a given point that possesses the following properties:

(i) the union of any collection of neighbourhoods is a neighbourhood;

(ii) the intersection of a finite number of neighbourhoods is a neighbourhood;

(iii) any set containing some neighbourhood $\Omega(x)$ is a neighbourhood of the point x.

THEOREM 3. *A subset $A\,(A \neq \varnothing)$ of a topological space (X, τ) is open if and only if it contains some neighbourhood of each of its points.*

PROOF. Let A be open, $x \in A$. It is clear then that A is a neighbourhood of x. Therefore, A contains a neighbourhood of any of its points. Let for any $x \in A$, there exist a neighbourhood of the point x, lying wholly in A. By the definition of a neighbourhood, it contains some open set U_x, $x \in U_x$. Consider the union $\bigcup_{x \in A} U_x$ of such sets for all $x \in A$. It is open; $A \subset \bigcup_{x \in A} U_x$ since any point of the set A belongs to $\bigcup_{x \in A} U_x$. On the other hand, we have: $U_x \subset A$ for every x, i.e., $\bigcup_{x \in A} U_x \subset A$. Therefore, $A = \bigcup_{x \in A} U_x$, and A is open. ■

Neighbourhoods are used for separating points from each other.

DEFINITION 5. A topological space (X, τ) is said to be *Hausdorff* if for any two different points x, y in it, there are neighbourhoods $U(x)$ and $U(y)$ of these points such that $U(x) \cap U(y) = \varnothing$.

A topological space (X, τ) equipped with the trivial topology is not Hausdorff if it contains more than one point (verify!).

These properties of the neighbourhoods of a point (which are now declared to be axioms) are often used as a basis for the following definition of a topology.

DEFINITION 6. A *topological space* is a set X for which each point x has a set of subsets $\{\Omega_\alpha(x)\}$, called the *neighbourhoods of the point* x satisfying the following conditions: (i) x belongs to each of its neighbourhoods $\Omega_\alpha(x)$; (ii) if a set $U \subset X$ con-

tains some $\Omega_\alpha(x)$ then U is also a neighbourhood of the point x; (iii) for any two neighbourhoods $\Omega_{\alpha_1}(x)$, $\Omega_{\alpha_2}(x)$ of the point x, their intersection $\Omega_{\alpha_1}(x) \cap \Omega_{\alpha_2}(x)$ is also a neighbourhood of the point x; (iv) for every neighbourhood $\Omega(x)$ of the point x, there is a neighbourhood $\Omega_{\alpha_1}(x) \subset \Omega(x)$ such that it is a neighbourhood of each of its points.

Exercise 3°. Show that sets which are neighbourhoods of each of their points and $\varnothing$ form a topology on X.

3. Continuous Mappings. Homeomorphisms.

We shall discuss now the definition of a continuous mapping of topological spaces.

Let (X, τ), (Y, σ) be two topological spaces endowed with topologies τ and σ, respectively. Let $f: X \to Y$ be a mapping of the sets.

DEFINITION 7. A mapping f of topological spaces is said to be *continuous* if the full inverse image $f^{-1}(V)$ of any open set V of the space (Y, σ) is an open set of the space (X, τ).

Exercises.

4°. State the definition of a continuous mapping in terms of a base and subbase for a topology.

5°. Show that a continuous numerical function $y = f(x)$ $(-\infty < x < +\infty)$ determines a continuous mapping $f: R^1 \to R^1$.

6°. Prove that f is continuous if and only if the inverse image $f^{-1}(F)$ is closed for any closed set F in Y.

If $f: X \to Y$, $g: X \to Z$ are mappings of topological spaces then it is natural to define the superposition $gf: X \to Z$ by the rule $(gf): x \to g(f(x))$.

THEOREM 4. *If f and g are continuous then gf is also continuous.*

The proof follows easily from the remark that

$$(gf)^{-1}(W) = f^{-1}(g^{-1}(W)),$$

where $W \subset Z$ is an arbitrary set.

We now state one of the most important definitions.

DEFINITION 8. Two topological spaces (X, τ), (Y, σ) are said to be *homeomorphic* if there exists a mapping $f: X \to Y$ that satisfies the conditions: (i) $f: X \to Y$ is a bijective mapping; (ii) f is continuous; (iii) f^{-1} is continuous.

Note that this definition exactly follows, in form, the definition of a homeomorphism of metric spaces.

DEFINITION 9. A mapping $f: X \to Y$ is said to be *open* (resp. *closed*) if the image of any open (resp. closed) set in X is open (resp. closed) in Y.

Exercise 7°. Prove that a mapping $f: X \to Y$ is a homeomorphism if and only if the mapping $f^{-1}: Y \to X$ is defined, and the mappings f and f^{-1} are both open and closed.

Thus, a homeomorphism transforms open sets into open, and closed sets into closed.

Associating each open set U of the space X with its image $f(U)$ under a homeomorphism $f: X \to Y$ establishes a bijective correspondence between the topologies on the spaces X and Y. Hence, any property of the space X stated in terms of a topology on this space is also valid for the space Y which is homeomorphic to X, and is similarly stated in terms of the topology on Y. Thus, the homeomorphic spaces X and Y possess identical properties and are indistinguishable from this point of view.

The properties of topological spaces that are preserved under homeomorphisms are called *topological properties**. Note, in this connection, that the main task of topology was for a long time (and still remains partially unsolved today) to discover an effective method of distinguishing between nonhomeomorphic spaces.

Exercises.

8°. Show that a homeomorphism determines a correspondence between the bases and subbases for homeomorphic spaces.

9°. Show that the homeomorphism relation is an equivalence relation.

10°. Show that the interval $(-1, +1)$ of the number line is homeomorphic to the whole number line and construct this homeomorphism.

11°. Show that a closed line-segment and an open interval on the number line are not homeomorphic.

There exists quite a useful extension of the notion of homeomorphism, viz., a *local homeomorphism*. This is a continuous mapping $f: X \to Y$ such that for any pair $x, y, y = f(x)$, there are neighbourhoods $U(x)$, $V(y)$ for which $f: U(x) \to V(y)$ is a homeomorphism.

Exercise 12°. Verify that the mapping $R^1 \setminus \{0\} \to R^1 \setminus \{0\}$ determined by the formula $y = x^2$ is a local homeomorphism.

4. A Subspace of a Topological Space. It can be seen from the above that subsets of metric and topological spaces are often considered independently. In addition, a subset Y of a metric space X naturally inherits the metric on X. We now define the notion of hereditary topology on Y if X is a topological space.

Let (X, τ) be a topological space, $Y \subset X$ a subset in X. Consider a family of subsets in Y:

$$\tau_Y = \{V : V = U \cap Y, U \in \tau\}.$$

THEOREM 5. *This family τ_Y is a topology on Y.*

THE PROOF is left to the reader (it is obvious).

The topology τ_Y is said to be an *induced* or *hereditary topology* from X and the space (Y, τ_Y) is called a *subspace* of the space (X, τ).

If $f: X \to Z$ is a continuous mapping of topological spaces and Y a subspace of X, then the mapping $f: Y \to Z$ can also be considered. This is called a *restriction* of f to Y and denoted by $f|_Y$.

* While investigating topological properties, homeomorphic spaces X and Y are often identified and written as $X = Y$. We will also follow this convention.

THEOREM 6. *The mapping $f|_Y : Y \to Z$ is continuous.*

PROOF. Let τ_z be a topology on the space Z, and $W \in \tau_z$. Then $(f|_Y)^{-1}(W) = f^{-1}(W) \cap Y$ and since $f^{-1}(W) \in \tau$ we will have $(f|_Y)^{-1}(W) \in \tau_Y$. ■

Exercises.

13°. Show that an open set in a subspace Y of a space X is not necessarily open in X. Consider the cases of $X = R^1, R^2, R^3$. Attempt the same question for closed sets in Y. Prove a priori that any closed set F_Y in Y is of the form $F_Y = F \cap Y$, where F is a closed set in X.

14°. Let $A, B \subset X$ be closed sets of a topological space X, and let $X = A \cup B$. Then a mapping $f: X \to Y$ is continuous if and only if $f|_A : A \to Y, f|_B : B \to Y$ are continuous.

We now introduce another important notion. A mapping $i: Y \to X$ is called an *embedding* of Y into X if (i) i is continuous, (ii) $i: Y \to i(Y)$ is a homeomorphism, where $i(Y) \subset X$ is a subspace of X.

Embeddings are useful when we intend to 'single out' a subspace $Y \subset X$ of the ambient space X and to consider it separately. The connection with X is preserved via the natural mapping $Y \to X$ which associates an element of Y with the same element of X and is an embedding.

2. TOPOLOGY AND CONTINUOUS MAPPINGS OF METRIC SPACES. SPACES R^n, S^{n-1} AND D^n

1. Topology in a Metric Space.

Let (X, ρ) be some metric space endowed with a metric ρ. A topology on it can be constructed in a natural manner. Consider all possible sets $D_\varepsilon(x) = \{y : \rho(y, x) < \varepsilon\}$, where $x \in X$, $\varepsilon > 0$. The set $D_\varepsilon(x)$ is called an *open ball* of radius ε with its centre at the point x.

The collection of all open balls $\{D_\varepsilon(x)\}$ forms a covering of the metric space for which the criterion of a base (Theorem 1, Sec. 1) is fulfilled. In fact, let $D_{\varepsilon_1}(x_1)$ and $D_{\varepsilon_2}(x_2)$ be two open balls whose intersection is nonempty. Let $y \in D_{\varepsilon_1}(x_1) \cap D_{\varepsilon_2}(x_2)$, $\delta = \min\{\varepsilon_1 - \rho(y, x_1), \varepsilon_2 - \rho(y, x_2)\}$, and let $z \in D_\delta(y)$; then

$$\rho(z, x_1) \leqslant \rho(z, y) + \rho(y, x_1) < \delta + \rho(y, x_1) \leqslant \varepsilon_1,$$
$$\rho(z, x_2) \leqslant \rho(z, y) + \rho(y, x_2) < \delta + \rho(y, x_2) \leqslant \varepsilon_2.$$

Therefore, $z \in D_{\varepsilon_1}(x_1) \cap D_{\varepsilon_2}(x_2)$, whence $D_\delta(y) \subset D_{\varepsilon_1}(x_1) \cap D_{\varepsilon_2}(x_2)$. Thus, the conditions of Theorem 1 are fulfilled.

DEFINITION 1. The topology τ_ρ determined by the base consisting of all open balls in a metric space (X, ρ) is called the topology *induced by the metric ρ*, or the *metric topology*.

Thus, open sets of the topology τ_ρ are all the possible unions of open balls of the metric space (X, ρ) (and $\varnothing$).

THEOREM 1. *The topology τ_ρ constructed is Hausdorff.*

PROOF. Let $x \neq y$. Then $\rho(x, y) = \alpha > 0$ (by a property of a metric). Setting $\varepsilon = \frac{\alpha}{3}$, we consider $D_\varepsilon(x)$, $D_\varepsilon(y)$. It is easy to show that $D_\varepsilon(x) \cap D_\varepsilon(y) = \varnothing$ In fact, if we assumed the contrary, we would have

$$\alpha = \rho(x, y) \leqslant \rho(x, z) + \rho(z, y) < \frac{\alpha}{3} + \frac{\alpha}{3} = \frac{2\alpha}{3}$$

for a point $z \in D_\varepsilon(x) \cap D_\varepsilon(y)$, which is impossible. ■

Another and equivalent definition of open sets in a metric space can be given.

DEFINITION 2. A set $U \neq \varnothing$ is *open* if for any $x \in U$, there is an open ball $D_\delta(x)$ with the centre at x which lies wholly in U.

Note that we defined a topology in R^2 in precisely the same manner, and, therefore, it coincides with the topology τ_ρ generated by the Euclidean metric ρ on the plane R^2. The verification of the equivalence of the two definitions is left to the reader.

Consider a mapping $f: X \to Y$ of a metric space (X, ρ_1) into a metric space (Y, ρ_2). Now two definitions of the continuity of the mapping f can be given, viz., as a mapping of metric and as a mapping of topological spaces. These two definitions are equivalent, viz., the following theorem is valid.

THEOREM 2. *A mapping $f: X \to Y$ of a metric space (X, ρ_1) into a metric space (Y, ρ_2) is continuous (for topologies induced by the metrics) if and only if for every $x_0 \in X$ and every sequence $\{x_n\}$ in X which converges to x_0, the sequence $\{f(x_n)\}$ converges to $f(x_0)$ in Y.*

PROOF. Let $f: X \to Y$ be a continuous mapping in topologies X, Y induced by the metrics, and let $x_n \xrightarrow{\rho_1} x_0$. We then show that $f(x_n) \xrightarrow{\rho_2} f(x_0)$, which means that for any $\varepsilon > 0$, there is a natural $N = N(\varepsilon, x_0)$ such that $\rho_2(f(x_n), f(x_0)) < \varepsilon$ when $n > N$.

Consider an open ball $D_\varepsilon(f(x_0))$ in Y and denote it by V_ε. Its inverse image $f^{-1}(V_\varepsilon)$ is an open set in X due to the continuity of f, moreover $x_0 \in f^{-1}(V_\varepsilon)$. The point x_0 belongs to $f^{-1}(V_\varepsilon)$ together with some ball $D_\delta(x_0)$ of radius δ. There exists a number $N(N = N(\varepsilon, x_0))$ such that x_n belongs to $D_\delta(x_0)$ (and to $f^{-1}(V_\varepsilon)$) when $n > N$, but then $f^{-1}(x_n) \in V_\varepsilon$ (i.e., $\rho_2(f(x_n), f(x_0)) < \varepsilon$) when $n > N$. Therefore, the mapping f is continuous as a mapping of metric spaces

Let the condition $f(x_n) \xrightarrow{\rho_2} f(x_0)$ be fulfilled for any sequence $\{x_n\}$ which is convergent to some point x_0 in the space X. We show then that the inverse image of any open set is open. Let V be an open set in Y, and $U = f^{-1}(V)$. We can show that U is open in the space X using the second definition of an open set. If $x \in f^{-1}(V)$, then it suffices to find $\varepsilon > 0$ such that $D_\varepsilon(x) \in f^{-1}(V)$. If we assume that such ε does not exist, then there exist sequences $\{\varepsilon_n\}$, $\{x_n\}$ such that $\varepsilon_n \to 0$, $x_n \in D_{\varepsilon_n}(x)$, but $x_n \bar{\in} f^{-1}(V)$. Therefore, $x_n \xrightarrow{\rho_1} x$, whence $f(x_n) \xrightarrow{\rho_2} f(x)$. Having noticed that $f(x)$

belongs to V together with a certain open ball, we conclude that $f(x_n) \in V$ and $x_n \in f^{-1}(V)$ beginning with some number n, which is contrary to the assumption. Thus, the mapping f is continuous in the topologies of the spaces X and Y which were induced by the metrics. ■

2. **Space R^n.** We shall consider an important example of a metric space, i.e., the *Euclidean space*

$$R^n = \{(\xi_1, \ldots, \xi_n), -\infty < \xi_i < +\infty, i = 1, \ldots, n\}$$

consisting of all ordered sets (called *points* or *vectors*) of n real numbers; the numbers ξ_i are called the *coordinates* of a point (vector).

A metric (the Euclidean metric) on $R^n (n \geqslant 1)$ is defined similarly to the metric on R^3:

$$\rho(x, y) = \left(\sum_{i=1}^{n} (\xi_i - \eta_i)^2 \right)^{1/2}, \tag{1}$$

where $x = (\xi_1, \ldots, \xi_n)$, $y = (\eta_1, \ldots, \eta_n)$ are two arbitrary vectors from R^n.

Let us verify that this is a metric. Evidently, Properties I, II, III of a metric (see Sec. 2, Ch. I) are fulfilled. Consider Property IV. It is required to prove the inequality

$$\left(\sum_{i=1}^{n} (\xi_i - \eta_i)^2 \right)^{1/2} \leqslant \left(\sum_{i=1}^{n} (\xi_i - \zeta_i)^2 \right)^{1/2} + \left(\sum_{i=1}^{n} (\zeta_i - \eta_i)^2 \right)^{1/2}$$

for arbitrary real numbers $\xi_i, \eta_i, \zeta_i, i = 1, \ldots, n$. The proof is broken into two lemmata.

LEMMA 1 (THE CAUCHY-BOUNIAKOWSKY INEQUALITY). *For any real numbers $\xi_i, \eta_i, i = 1, \ldots, n$, the following inequality holds*

$$\sum_{i=1}^{n} (\xi_i \eta_i) \leqslant \left(\sum_{i=1}^{n} \xi_i^2 \right)^{1/2} \left(\sum_{i=1}^{n} \eta_i^2 \right)^{1/2}.$$

PROOF. For an arbitrary real λ, we have $\sum_{i=1}^{n} (\xi_i + \lambda \eta_i)^2 \geqslant 0$, whence $\sum_{i=1}^{n} \xi_i^2 + 2\lambda \sum_{i=1}^{n} \xi_i \eta_i + \lambda^2 \sum_{i=1}^{n} \eta_i^2 \geqslant 0$. Consider the left-hand side of the inequality as a polynomial in λ. It cannot have two different real roots. Therefore, its discriminant is non-positive. Hence, the inequality

$$\left(\sum_{i=1}^{n} \xi_i \eta_i \right)^2 \leqslant \sum_{i=1}^{n} \xi_i^2 \sum_{i=1}^{n} \eta_i^2. \ ■$$

LEMMA 2 (THE MINKOWSKI INEQUALITY). *For arbitrary real numbers* ξ_i, η_i, $i = 1, \ldots, n$, *the following inequality is valid*

$$\left(\sum_{i=1}^{n} (\xi_i + \eta_i)^2 \right)^{1/2} \leqslant \left(\sum_{i=1}^{n} \xi_i^2 \right)^{1/2} + \left(\sum_{i=1}^{n} \eta_i^2 \right)^{1/2}.$$

PROOF. By using the Cauchy-Bouniakowsky inequality,

$$\sum_{i=1}^{n} (\xi_i + \eta_i)^2 = \sum_{i=1}^{n} (\xi_i^2 + 2\xi_i\eta_i + \eta_i^2)$$

$$\leqslant \sum_{i=1}^{n} \xi_i^2 + 2 \left(\sum_{i=1}^{n} \xi_i^2 \right)^{1/2} \left(\sum_{i=1}^{n} \eta_i^2 \right)^{1/2} + \sum_{i=1}^{n} \eta_i^2$$

$$= \left[\left(\sum_{i=1}^{n} \xi_i^2 \right)^{1/2} + \left(\sum_{i=1}^{n} \eta_i^2 \right)^{1/2} \right]^2,$$

and by taking the square root of both sides of this inequality, we obtain the required inequality. ■

We can now complete the verification of Property IV of the metric. Using the Minkowski inequality, we obtain

$$\left(\sum_{i=1}^{n} (\xi_i - \eta_i)^2 \right)^{1/2} = \left(\sum_{i=1}^{n} [(\xi_i - \zeta_i) + (\zeta_i - \eta_i)]^2 \right)^{1/2}$$

$$\leqslant \left(\sum_{i=1}^{n} (\xi_i - \zeta_i)^2 \right)^{1/2} + \left(\sum_{i=1}^{n} (\zeta_i - \eta_i)^2 \right)^{1/2}$$

Thus, ρ is a metric on R^n. ■

Let $x_0 = (\xi_1^0, \ldots, \xi_n^0)$ be the centre of a ball $D^n(x_0)$, and $x = (\xi_1, \ldots, \xi_n)$ its arbitrary point. Then the coordinates of a point x satisfy the inequality

$$|\xi_1 - \xi_1^0|^2 + \ldots + |\xi_n - \xi_n^0|^2 < r^2. \tag{2}$$

A ball in R^n is often denoted by $D_r^n(x_0)$ and called an *open n-disc*. A set of points x whose coordinates satisfy the unstrict inequality

$$|\xi_i - \xi_1^0|^2 + \ldots + |\xi_n - \zeta_n^0|^2 \leqslant r^2 \tag{3}$$

is called a *closed ball* (*closed n-disc*) $\overline{D}_r^n(x_0)$. The $(n - 1)$-dimensional sphere $S_r^{n-1}(x_0)$ with radius r and centre at the point x_0 is defined by the equality

$$|\xi_1 - \xi_1^0|^2 + \ldots + |\xi_n - \xi_n^0|^2 = r^2. \tag{4}$$

We will call it the *boundary of the disc* $\overline{D}_r^n$ or D_r^n.

A metric on R^n can be defined in other ways, for example,

$$\rho(x, y) = \max_{i = 1, \ldots, n} \{|\xi_i - \eta_i|\}. \tag{5}$$

Exercise 1°. Describe a ball in R^n by means of metric (5). Show that the Euclidean metric and metric (5) induce the same topology.

Consider the complex n-dimensional space C^n:

$$C^n = \{z : z = (z_1, \ldots, z_n), z_k = x_k + iy_k, x_k, y_k \in R^1, k = 1, \ldots, n\}.$$

The metric on it is introduced in the same way as in the real case:

$$\rho(z', z'') = (|z_1' - z_1''|^2 + \ldots + |z_n' - z_n''|^2)^{1/2},$$

where $z' = (z_1', \ldots, z_n')$, $z'' = (z_1'', \ldots, z_n'')$ are elements of C^n. The same topology is determined by the metric

$$\rho(z', z'') = \max_{k=1, \ldots, n} |z_k' - z_k''|.$$

We now formulate a condition for the continuity of mappings of Euclidean spaces. A mapping $f: R^n \to R^m$ associates each point $(\xi_1, \ldots, \xi_n)$ with a certain point $(\eta_1, \ldots, \eta_n)$, so that we can write

$$\begin{aligned} \eta_1 &= f_1(\xi_1, \ldots, \xi_n), \\ &\ldots\ldots\ldots\ldots \\ \eta_m &= f_n(\xi_1, \ldots, \xi_n), \end{aligned} \tag{6}$$

where f_i, $i = 1, \ldots, m$ is a numerical function of n variables. This function determines a mapping f_i: $R^n \to R^1$ by the rule

$$\eta_i = f_i(\xi_1, \ldots, \xi_n). \tag{7}$$

It is evident that the continuity of the mapping f_i is equivalent to the continuity of the numerical function $f_i(\xi_1, \ldots, \xi_n)$ as it is defined in analysis.

Call mapping (7) the i-th component of the mapping f. The mapping f is determined by specifying all its components f_i, $i = 1, \ldots, m$.

THEOREM 3. *A mapping f: $R^n \to R^m$ is continuous if and only if each of its components $f_i : R^n \to R^1$, $i = 1, \ldots, m$, is continuous.*

The proof follows from the remark that $f(x^k) \to f(x^0)$, $k \to \infty$, is equivalent to $f_i(x^k) \to f_i(x^0)$, $k \to \infty$, for $i = 1, \ldots, m$.

3. **The Ball D^m is Homeomorphic to R^m.** Consider some subsets of R^n, $n \geqslant 2$. Let S^{n-1} be a sphere, and D^n an open n-disc with unit radius and centre at the point $(0, \ldots, 0)$. Denote the part of the sphere where $\xi_n > 0$ (i.e., the northern hemisphere) by S_+^{n-1}. We prove that the disc D^{n-1} is homeomorphic to the hemisphere S_+^{n-1}.

The space R^{n-1} may be considered to be coincident with the subspace of points $(\xi_1, \ldots, \xi_{n-1}, 0)$ of the space R^n if the points $(\xi_1, \ldots, \xi_{n-1})$ and $(\xi_1, \ldots, \xi_{n-1}, 0)$ are identified. Then D^{n-1} and S_+^{n-1} lie in R^n and are given thus:

$$\begin{aligned} S_+^{n-1} &= \{(\xi_1, \ldots, \xi_n): \xi_i^2 + \ldots + \xi_n^2 = 1, \xi_n > 0\}, \\ D^{n-1} &= \{(\xi_1, \ldots, \xi_n): \xi_1^2 + \ldots + \xi_{n-1}^2 < 1, \xi_n = 0\}. \end{aligned}$$

In the case of R^3, we have: S_+^2 is the upper half of the sphere without the equator,

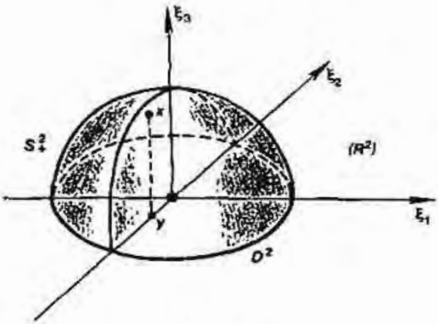


Fig. 32

and D^2 is the interior of the unit circle in R^2 (Fig. 32),

$$x = (\xi_1, \xi_2, \xi_3) \in S^2{}_+ , y = (\xi_1, \xi_2, 0) \in D^2.$$

The projection $(\xi_1, \xi_2, \xi_3) \to (\xi_1, \xi_2, 0)$ is a homeomorphism of the hemisphere S^2_+ and disc D^2

Similarly, the projection

$$f : (\xi_1, \ldots, \xi_{n-1}, \xi_n) \to (\xi_1, \ldots, \xi_{n-1}, 0)$$

determines a continuous bijective mapping of S^{n-1}_+ onto D^{n-1} in R^n (verify!). Consider the inverse mapping. It is easy to see that it is of the form

$$f^{-1} : (\xi_1, \ldots, \xi_{n-1}, 0) \to (\xi_1, \ldots, \xi_{n-1}, (1 - \xi_1^2 - \ldots - \xi_{n-1}^2)^{1/2}) \qquad (8)$$

and continuous. Thus, a homeomorphism of the disc D^n and hemisphere S^{n-1}_+ has been constructed. Call the set of points of the sphere S^{n-1} that satisfy the inequality $\xi_n \geqslant 0$ the *closed hemisphere* $\overline{S}^{n-1}_+$. It is clear that $\overline{S}^{n-1}_+ = S^{n-1}_+ \cup S^{n-2}$. It is natural to call the sphere S^{n-2} the *boundary of the hemisphere* $\overline{S}^{n-1}_+$ (or S^{n-1}_+). Note that S^{n-2} is simultaneously the boundary of the disc $\overline{D}^{n-1}$ (or D^{n-1}). It is easy to see that homeomorphism (8) is also defined on S^{n-2}, and that $f^{-1}|_{S^{n-2}} = 1_{S^{n-2}}$. Thus, $\overline{D}^{n-1}$ is homeomorphic to $\overline{S}^{n-1}_+$.

We now establish another important homeomorphism.

THEOREM 4. *The disc D^m is homeomorphic to the space R^m, $m \geqslant 1$.*

PROOF. Putting $m = n - 1$, we use the previous construction. We translate the space R^{n-1}, $n \geqslant 2$, so that the origin of coordinates goes to the point $(0, \ldots, 0, 1)$, the North Pole of the sphere S^{n-1}. Every point in the new plane has the form $(\xi_1, \xi_2, \ldots, \xi_{n-1}, 1)$. If we draw the half-line $\eta_1 = t\xi_1, \eta_2 = t\xi_2, \ldots, \eta_n = t\xi_n$, $t \geqslant 0$ through each point $x = (\xi_1, \ldots, \xi_n) \in S^{n-1}$, it will intersect the constructed plane at a unique point corresponding to the value $t(x) = 1/\xi_n$. By assign-

ing this intersection point to the point x, we obtain the mapping $\Phi: S_+^{n-1} \to R^{n-1}$ given by the rule

$$(\xi_1, \ldots, \xi_n) \to \left(\frac{\xi_1}{\xi_n}, \ldots, \frac{\xi_{n-1}}{\xi_n}, 1\right).$$

This mapping, as it is easy to verify, is a homeomorphism. The superposition of the homeomorphisms

$$\Phi f^{-1}: D^{n-1} \to R^{n-1}, n \geqslant 2,$$

yields the required homeomorphism. ■

Exercises.

2°. State a criterion of the continuity of a mapping $f: C^n \to C^m$ of complex spaces.
3°. Prove that C^n is homeomorphic to R^{2n}.
4°. Prove that the balls in the space R^n which are defined using metrics (1) and (5) are homeomorphic.
5°. Prove the continuity of the functions

$$f(\xi_1, \xi_2) = (\xi_1^2 + \xi_2^2)^{1/2}, f(\xi_1, \ldots, \xi_n) = (\xi_1^2 + \ldots + \xi_n^2)^{1/2}.$$

6°. Define discs and a sphere in the space C^n by conditions (2)-(4) and denote them by $D^n_{C,r}$, $\widetilde{D}^n_{C,r}$ and $S^{n-1}_{C,r}$, respectively. Prove that they are homeomorphic to D^{2n}_r, $\widetilde{D}^{2n}_r$ and S^{2n-1}_r, respectively.
7°. Prove that discs of any radii are homeomorphic in R^n; prove a similar statement for spheres.

3. FACTOR SPACE AND QUOTIENT TOPOLOGY

1. The Definition of a Quotient Topology. We will give a strict definition of a topology in a factor space, i.e., a quotient topology, and analyse the examples from Sec. 3, Ch. 1, from the new point of view. Let a relation $x \sim y$ between some elements $x, y \in X$ be defined on an abstract set X. This relation is called an *equivalence* if the following properties are fulfilled: (1) $x \sim x$ for any $x \in X$ (reflexivity); (2) if $x \sim y$ then $y \sim x$ (symmetry); (3) if $x \sim y$ and $y \sim z$ then $x \sim z$ (transitivity).

The set X is then split into disjoint classes of equivalent elements, or *equivalence classes*.

The set $\{D_\alpha\}$ of all equivalence classes will be denoted by X/R, where R denotes the equivalence in X.

DEFINITION. The set X/R is called the *factor set* of the set X with respect to the equivalence R.

Let (X, τ) be a topological space, and let an equivalence relation R be defined on the set X. A natural topology may then be introduced into the factor set X/R in the following manner: we call a subset $V \subset \{D_\alpha\}$ consisting of elements of D_α open if and only if the union $\cup D_\alpha$ of the sets D_α as subsets of X is open in the space (X, τ).

Naturally, we refer to the empty set as an open set. This collection of open subsets in X/R is a topology and denoted by τ_R.

Exercise 1°. Verify that τ_R is a topology on X/R.

The topology τ_R is called a quotient topology, and is usually implied when a factor space is being spoken of.

The motives for defining the topology τ_R will become clearer if the mapping $\pi : X \to X/R$ associating every element $x \in X$ with the equivalence class D_x is considered. This mapping is called the *projection* of the space X onto the factor space. It is easy to see that a set $V \subset X/R$ is open if and only if the set $\pi^{-1}(V)$ is open in X. Thus, the projection π is continuous as a mapping from (X, τ) to $(X/R, \tau_R)$. (Note that this entails the principle of the continuity of the 'gluing' mapping which we mentioned in Sec. 3, Ch. I).

There may certainly exist other topologies on the set X/R in which the projection π is continuous. The following theorem characterizes the topology τ_R.

THEOREM 1. *The topology τ_R is the strongest of all topologies on X/R for which the mapping π is continuous.*

PROOF. If $[W]$ is a topology on X/R in which the mapping π is continuous, then $\pi^{-1}(W)$ is open in X. Therefore, W is open in the factor space X/R, i.e., $W \in \tau_R$. This means that the topology $[W]$ is weaker than the topology τ_R. ■

Exercise 2°. Let $X = [0, 1] \subset R^1$. We define the equivalence thus: $x \overset{R}{\sim} y \Leftrightarrow x - y$ is rational. Show that the factor space X/R is not Hausdorff.

2. Examples of Factor Spaces.

Consider the examples of Sec. 3, Ch. I. If X is a rectangle $abcd$, and an equivalence relation R is defined so that $x \sim x$ for each $x \in X$ and $x \sim y$ if and only if $x \in ab$, $y \in cd$ and x, y lie on the same horizontal in X, then X/R is a topological space which is homeomorphic to the cylinder (see Figs. 1 and 2).

In fact, the base for the topology of the cylinder is formed by two-dimensional 'discs', i.e., the intersections of balls in R^3 with the cylinder (Fig. 33). If the cylinder is cut along the line ab and developed into a rectangle then the 'discs' will be carried into the base for the topology of the latter. Moreover, the 'discs' intersecting the

Fig. 33

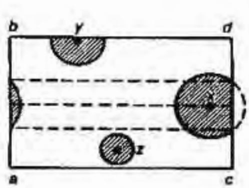


Fig. 34

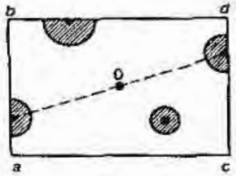


Fig. 35

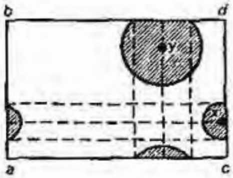


Fig. 36

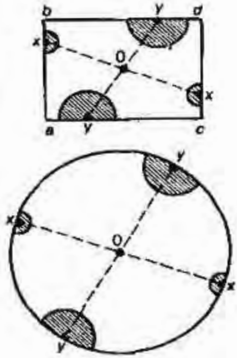


Fig. 37

line *ab* will be cut into segments which complement each other to circles and lie on the opposite sides of the rectangle. It is clear therefore that it is necessary to paste the complementary segments together along the line of the cut in order to obtain the base for the topology in X/R (Fig. 34). It is now easy to see that by associating equivalent points of the rectangle with the point into which they have been 'glued', we obtain a homeomorphism of our factor space X/R with the cylinder.

The topology of a Möbius strip can be investigated in precisely the same way (see the next example of the 'gluing' in Sec. 3, Ch. I). Some open sets of a Möbius strip are represented in Fig. 35. Here, the segments 'glued together' consist of points that are symmetric with respect to the centre and lie on the sides *ab* and *cd*.

In the third example of 'gluing', the corresponding factor space is homeomorphic to the torus; the elements of the base for its topology are represented in Fig. 36. Here, the corresponding segments are not only glued together along the vertical bases lying on *ab*, *cd*, but also along the horizontal bases lying on *ac*, *bd*.

Finally, in the last example, we obtain a projective plane. Elements of the base for its topology are represented in Fig. 37. Here, the segments are glued together along both vertical and horizontal boundaries of the rectangle by joining points which are symmetrical with respect to the centre.

Here is another useful example of how to form factor spaces. Let $Y \subset X$ be a subspace of a topological space X. We declare every point of Y to be equivalent to every other point, and all the points $x \in X \setminus Y$ equivalent to themselves. The factor space with respect to this equivalence is denoted by X/Y, and the projection $\pi : X \to X/Y$ is called *shrinking the set* Y to a point. E.g., $S^1 = I/\{0, 1\}$ is the factor space of the line-segment $I = [0, 1]$ with respect to the set of end-points.

3. **Mappings of Factor Spaces.** Let X, X' be two topological spaces and R, R' equivalences on them. Consider a mapping $f: X \to X'$. We will say that the mapping f *preserves equivalence* if it follows from $x \overset{R}{\sim} y$ that $f(x) \overset{R'}{\sim} f(y)$. For such mappings, it is natural to define the mapping $\hat{f}: X/R \to X'/R'$ of the factor spaces as follows: let D_α be an equivalence class in X, $x \in D_\alpha$ any element and D'_β the equivalence class in X' containing the point $f(x)$, then $\hat{f}(D_\alpha) = D'_\beta$.

Exercise 3°. Show that the definition of $\hat{f}$ is valid. Note that the mapping $\hat{f}$ is called a *residue class mapping*.

THEOREM 2. *If a continuous mapping $f: X \to X'$ preserves equivalence then the corresponding residue class mapping $\hat{f}: X/R \to X'/R'$ is continuous.*

PROOF. Denote the projections of the spaces X, X' onto the corresponding factor spaces by π, π', respectively. The diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X' \arrow[d, "\pi'"] \\ X/R \arrow[r, "\hat{f}"] & X'/R' \end{tikzcd}$$

is commutative, i.e., for each $x \in X$ we have $(\hat{f}\pi)(x) = (\pi' f)(x)$. If a set V is open in X'/R' then $(\pi' f)^{-1}(V)$ is open in X, because $\pi' f$ is continuous. But $(\hat{f}\pi)^{-1}(V) = (\pi' f)^{-1}(V)$ and therefore the set $(\hat{f}\pi)^{-1}(V)$ is open in X. Since $(\hat{f}\pi)^{-1}(V) = \pi^{-1}(\hat{f}^{-1}(V))$, the set $\hat{f}^{-1}(V)$ is open in X/R (by the definition of the topology of a factor space). ■

We now formulate a test to see if factor spaces are homeomorphic.

THEOREM 3. *If $f: X \to X'$ is a homeomorphism and the mappings f, f^{-1} preserve equivalence then the residue class mapping $\hat{f}: X/R \to X'/R'$ is a homeomorphism.*

In fact, the mapping f^{-1} in this case determines the residue class mapping $\widehat{f^{-1}} = (\hat{f})^{-1}$ (verify!), and Theorem 2 can be applied both to $\hat{f}$ and $(\hat{f})^{-1}$.

Here are three more 'models' of the projective plane RP^2 in addition to those listed in Sec. 3, Ch. I. The first is obtained from the sphere $X = S^2$ by pasting together diametrically opposite points (Fig. 38). The second consists of straight lines in R^3 that pass through zero ($x \overset{R}{\sim} y \Leftrightarrow x, y$ lie on one such straight line and $x \neq 0, y \neq 0$) (Fig. 38).

Exercise 4°. Describe the topology of the spaces obtained as the topology of the factor spaces S^2/R and $(R^3/0)/R$, respectively.

The third model of RP^2 can be described as follows. Consider an arbitrary plane P in R^3 which does not pass through the origin. Fix the point a, i.e., the projection of the origin of R^3 onto P. In accordance with the second model of RP^2 just considered, this space consists of the straight lines in R^3 that pass through the origin. Associate each of these straight lines with the point where it intersects the plane P if it intersects P, or with the straight line in P passing through a and parallel to the one given if not. The straight line obtained on the plane P is symbolically identified with a point at infinity where these parallel lines meet.

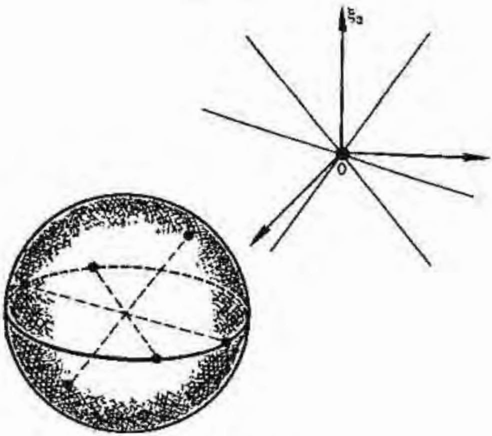


Fig. 38

Thus, we obtained a one-to-one correspondence between RP^2 (second model) and P with the points at infinity added to it, one point for each direction (i.e., for each straight line passing through the origin) in P. In the set obtained, the plane P is considered to be endowed with the usual topology: a *neighbourhood of a point at infinity* corresponding to a direction d on P is defined to be a part of the plane P (the shaded region in Fig. 39) bounded by an arbitrary hyperbola with the axis d. The set of all points at infinity added to the plane P is also called the *absolute* or *straight line at infinity*.

Exercise 5°. Prove the homeomorphism of all realizations of RP^2.

Consider a closed disc $\bar{D}^n$ and its boundary S^{n-1}. We identify all boundary points. Denote the factor space obtained by $\bar{D}^n/S^{n-1}$

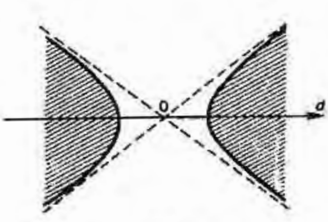


Fig. 39

THEOREM 4. *The space $\bar{D}^n/S^{n-1}$ is homeomorphic to the sphere S^n.*

PROOF. In Item 3, Sec. 2, it was shown that the disc D^n is homeomorphic to the closed hemisphere $\bar{S}^n_+$. This homeomorphism is the identity homeomorphism on the common boundary (S^{n-1}) of these sets. Therefore, the equivalence relation from D^n is induced on $\bar{S}^n_+$, and $\bar{D}^n/S^{n-1}$ is homeomorphic to $\bar{S}^n_+/S^{n-1}$ by the last theorem.

We shall now show that S^n_+/S^{n-1} is homeomorphic to S^n. The inclusion $\bar{S}^n_+ \subset S^n$ is natural. Denote the South Pole $(0, 0, \ldots, 0, -1)$ of the sphere S^n by $*$. There must therefore exist a continuous surjective mapping $\varphi: \bar{S}^n_+ \to S^n$ such that $\varphi(S^{n-1}) = *$, and $\varphi|S^n_+: S^n_+ \to S^n \setminus \{*\}$ is a homeomorphism. The latter can be constructed, for example, as follows: if $x \in \bar{S}^n_+$ and $x \neq N$ (N is the North Pole) then we draw a two-dimensional plane through the points $0, N, x$ which intersects S^n along a circumference (meridian). Shifting x along the meridian through an arc that is twice the arc xN, we obtain the point $\varphi(x)$. Put $\varphi(N) = N$. The residue class mapping is thus defined

$$\hat{\varphi}: \bar{S}^n_+/S^{n-1} \to S^n/\{*\} = S^n$$

which is evidently a homeomorphism.

The product of the two homeomorphisms

$$\bar{D}^n/S^{n-1} \to \bar{S}^n_+/S^{n-1}, \quad \bar{S}^n_+/S^{n-1} \to S^n,$$

is the required homeomorphism. ■

4. CLASSIFICATION OF SURFACES

1. Surfaces and Their Triangulation. Let us return to our investigation of closed surfaces. The definitions of a topological space, factor space, homeomorphism of topological spaces given above and the examples considered make up a solid basis for the proof of the theorem mentioned in Sec. 3, Ch. I. It states that any closed surface is topologically equivalent to an M_p- or N_q- surface, i.e., to a sphere with p handles or q Möbius strips glued to it. Here, the corresponding notions will be made more precise, and the proof of the above-mentioned theorem given.

A topological space X each of whose points has a neighbourhood homeomorphic to the open two-dimensional disc will be called a *two-dimensional manifold*. It is more convenient to study these spaces if they are broken into elementary pieces which are topologically equivalent to triangles in the two-dimensional Euclidean plane. Let us make this representation more precise.

DEFINITION 1. Call the pair (T, φ), where T is a subspace in X, and $\varphi: \Delta \to T$ is a homeomorphism of some triangle $\Delta \subset R^2$ on T, a *topological triangle* in X.

If the homeomorphism $\varphi: \Delta \to T$ is fixed (when this cannot be a cause of ambiguity), then the subspace $T \subset X$ will be called, for short, a topological triangle. The images of the vertices and the sides of the triangle Δ (along with a restriction of the homeomorphism φ) are called, respectively, the *vertices*, and the *edges* of the

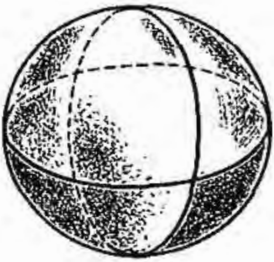


Fig. 40

topological triangle T. For uniformity, it is also convenient to call the sides of the triangle Δ the edges.

We now define an orientation of a triangle. Different ordered triples of points can be formed from the vertices of Δ. We consider two triples to be equivalent if one can be obtained from the other by a cyclic permutation. Clearly, there are exactly two equivalence classes. A *triangle* Δ *is oriented* if one of these equivalence classes is fixed. A *topological triangle* (T, φ) is said to be *oriented* if the triangle Δ is oriented. It is obvious that orienting a triangle Δ is equivalent to giving a certain direction to the circumnavigation of its vertices (clockwise or counterclockwise). This circumnavigation direction determines, via the homeomorphism φ, a direction of the circumnavigation of the vertices of the topological triangle, i.e., an orientation induced by the homeomorphism φ. An orientation of a triangle obviously determines the orientations of its edges (i.e., ordered pairs of its vertices).

Note for the future that an orientation of an n-gon and its edges when $n > 3$ is defined in precisely the same way (by giving an orientation for the circumnavigation of the vertices).

DEFINITION 2. A finite set $K = \{(T_i, \varphi_i)\}_{i=1}^{k}$ of topological triangles in X that fulfil the conditions (1) $X = \bigcup_{i=1}^{k} T_i$ and (2) the intersection of any pair of triangles from K is either empty or coincides with their common vertex or common edge, is called a *triangulation* of the two-dimensional manifold X.

A manifold for which there exists a triangulation is said to be *triangulable*. If any two vertices of the triangles from K can be joined by a path made up of the edges, then we call X *connected*.

In Fig. 40, an example of a triangulation of the sphere S^2 consisting of eight triangles is shown.

DEFINITION 3. We call a connected, triangulable, two-dimensional manifold a *closed surface*.

Note that the examples of closed surfaces which can be triangulated into topological polygons and which we considered in Sec. 3, Ch. 1, are examples of closed surfaces in the sense of Definition 3 (to prove this, it suffices to triangulate the polygons).

Exercise 1°. Construct triangulations of a torus and a projective plane. Verify that they are closed surfaces.

The topological properties of a closed surface are determined by the structure of its triangulation. To investigate the triangulation, it is convenient to consider its schematic representation on the plane. Moreover, plane triangles Δ_i and the inverse images of the triangles $T_i \in K$ may be considered to belong to the same plane and be mutually exclusive.

We will describe such a representation. Let (T_i, φ_i), (T_j, φ_j) be two triangles from K, and $T_i \cap T_j = a$ their common edge; let $a_i = \varphi_i^{-1}(a)$ and $a_j = \varphi_j^{-1}(a)$ be the corresponding edges in Δ_i, Δ_j. The gluing homeomorphism is defined as

$$\varphi_{ij} = \varphi_j|_a^{-1}\varphi_i|_{a_i} : a_i \to a_j.$$

Thus, the triangulation K can be associated with a set $\Delta = (\{\Delta_i\}_1^k = 1, \{\varphi_{ij}\})$ of triangles of the plane along with the homeomorphisms φ_{ij} for the corresponding pairs of the edges. We declare that points in $\bigcup\limits_{i=1}^{k} \Delta_i$ that correspond to each other under the homeomorphisms φ_{ij} are equivalent. We will denote this equivalence by R.

LEMMA. *The factor space* $\left(\bigcup\limits_{i=1}^{k} \Delta_i\right)/R$ *is homeomorphic to the surface* X.

PROOF. The homeomorphisms φ_i: $\Delta_i \to T_i$ naturally determine the surjective mapping Φ: $\bigcup\limits_{i=1}^{k} \Delta_i \to X$, the inverse image $\Phi^{-1}(x)$ for any $x \in X$ being an R-equivalence class. The residue class mapping $\hat{\Phi}$: $\left(\bigcup\limits_{i=1}^{k} \Delta_i\right)/R \to X$ is a continuous mapping by Theorem 2 of Sec. 3. Obviously, it is bijective and its inverse $\hat{\Phi}^{-1}$ continuous. ■

2. The Development of a Surface.

We shall need later some systems which are analogous to the family Δ schematically representing a triangulation K of the surface X, but such that along with triangles they may contain n-gons $(n > 3)$.

DEFINITION 4. A family $Q = (\{Q_i\}, \{\varphi_{ij}\})$, where $\{Q_i\}$ is a finite set of disjoint plane polygons and $\{\varphi_{ij}\}$ a finite set of gluing homeomorphisms of pairs of edges, each edge being glued to only one edge, is called a *development*. Gluing the edges of the same polygon together is permitted.

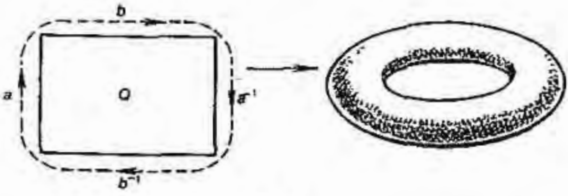


Fig. 41

Note that if the the location of a polygon Q_i on the plane is altered by a homeomorphism α_i, then we get new homeomorphisms $\{\alpha_j \varphi_{ij} \alpha_i^{-1}\}$ that glue its edges, and which we shall not distinguish hereafter from the homeomorphisms $\{\varphi_{ij}\}$. In particular, the family $\Delta = (\{\Delta_i\}, \{\varphi_{ij}\})$ is a development. The development Δ is said to agree with the triangulation K.

Consider the factor-space $\hat{Q}$ of the union $\bigcup_i Q_i$ with respect to the equivalence R determined by homeomorphisms $\{\varphi_{ij}\}$, $\hat{Q} = \left(\bigcup Q_i\right)/R$ for an arbitrary development Q. We will call $\hat{Q}$ the factor space of the development Q. It is clear that the factor space of a development is a two-dimensional manifold; it admits a triangulation generated by a sufficiently fine triangulation of the polygons Q_i. Thus, if the factor space $\hat{Q}$ is connected then it is a closed surface (hereafter, only such $\hat{Q}$ are considered). We will call Q the *development of the surface* $\hat{Q}$ in such a case.

A residue class mapping induces a decomposition of the surface $\hat{Q}$ into the images of polygons, images of edges (or the decomposition edges), images of vertices (or the decomposition vertices), the decomposition not being, generally speaking, a triangulation.

In Fig. 41, there is the development of a torus represented as a polygon. The arrows and designations of its edges denote the rule for gluing the torus.

Hereafter, we will orient development polygons by fixing an orientation for each of them. The orientations of the polygons determine the corresponding orientations of the edges. Under the gluing homeomorphism $\varphi_{ij}: a_i - a_j$ of two edges, the edge a_j acquires an orientation induced (from the orientation of the edge a_i) by the homeomorphism φ_{ij}. Generally speaking, this orientation may be different from the orientation of the edge a_j.

A development Q is said to be *orientable* if for the same orientation of all its polygons (e.g., when circumnavigating the vertices counterclockwise), the edge gluing homeomorphisms induce the reverse orientation in the image edge. Otherwise (i.e., if the original orientation coincides with the induced in at least one edge), the development is said to be *non-orientable*.

A surface X is said to be *orientable* (resp. *non-orientable*) depending upon whether its development is orientable (resp. non-orientable).

3. The Classification of Developments.

DEFINITION 5. Two developments Q and Q^* are said to be *equivalent* if their factor spaces are homeomorphic.

We now introduce some elementary operations over a development which will transform it into an equivalent one.

SUBDIVISION. Let there be an n-gon Q_i $(n > 3)$ in a development. We draw a diagonal d which breaks Q_i into two polygons Q_i' and Q_i'' and move the polygons Q_i' and Q_i'' apart constructing a new development $\tilde{Q}$ from Q by replacing the polygon Q_i by two polygons Q_i', Q_i''. The two new edges d' and d'', i.e., replicas of the diagonal d, are related by the natural identity homeomorphism, while the homeomorphisms of the original edges are preserved. The development $\tilde{Q}$ is called a *subdivision of the development* Q; it is obvious that they are equivalent.

GLUING. This operation is inverse to the last one. Two polygons Q_i' and Q_i'' of a development Q are glued into one polygon Q_i under one of the homeomorphisms of their edges d' and d''; the homeomorphisms of the remaining edges Q_i' and Q_i'' induce homeomorphisms of the edges of the polygon Q_i.

CONVOLUTION. Let two adjacent edges in a polygon Q_i with opposite orientations be glued together. After they have been 'glued', we obtain a development which, instead of Q_i, contains a polygon in which the number of vertices is two less, and the number of homeomorphisms is one less (Fig. 42).

We stress that the described operations preserve the development equivalence class. We leave the proof to the reader.

For convenience, in the theory to come, we will describe each development in symbolic words as follows: given an arbitrary development $Q = (\{Q_i\}, \{\varphi_{ij}\})$, we specify a circumnavigation (i.e., orientation) for each polygon. Each edge of each polygon Q_i will be denoted by a letter according to the following rule: given a homeomorphism φ_{ij} for a pair of edges, we denote one of the edges by a and check whether the orientation that the homeomorphism φ_{ij} induces (carries over) from a onto the second edge coincides with the orientation of the latter. If they do coincide then the second edge of the pair is also denoted by a; if not it is denoted by a^{-1}. Edges which are not glued to a are denoted by other letters, e.g., b, c or b^{-1}, c^{-1},

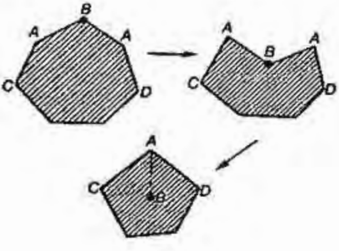


Fig. 42

etc. See Ch. I for the words of the sides of the pentagon and triangle which are the developments of a handle and Möbius strip, respectively.

Having denoted the edges of all the polygons Q_i, we obtain a set of words $\{\omega(Q_i)\}$, where $\omega(Q_i)$ is a word denoting the 'gluing' rule of the polygon Q_i. In addition, the letters in the word $\omega(Q_i)$ are written in the order in which are the corresponding sides of the polygon Q_i according to its orientation. It is clear that the indicated set of symbolic words $\{\omega(Q_i)\}$ determines the development Q.

Two main types of developments can be singled out.

DEFINITION 6. A *Type I canonical development* is a development consisting of one polygon determined by a word whose form is aa^{-1} or

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\ldots a_mb_ma_m^{-1}b_m^{-1},\ m > 0.$$

DEFINITION 7. A *Type II canonical development* is a development consisting of one polygon with a word of the form $a_1a_1a_2a_2\ldots a_ma_m$, $m > 0$.

We now formulate the basic result.

THEOREM 1. *Any development is equivalent to a Type I or II canonical development according to its orientability or non-orientability.*

PROOF Two remarks at first. To begin with, it is easy to see that by gluing, the development corresponding to a triangulation K of a surface X can be reduced to a development consisting of one polygon. We shall therefore consider only this kind of development. Secondly, if there are combinations of the form aa^{-1} in a word other than aa^{-1} in our development, then we may get rid of them by the convolution of the edges a and a^{-1} around their common vertex A. The word of the new development is derived from the original by crossing out all combinations aa^{-1}.

Finally, we come either to a two-letter word (aa^{-1} or aa) or to a word no less than four letters long and without combinations of the form aa^{-1} (recall that the surface is closed). Since the words aa, aa^{-1} describe a canonical development, only the last case should be considered further.

We shall break this analysis into a number of steps:

(1) The obtained development Q' can be transformed into one whose all vertices are equivalent, i.e., glued on factorization. In fact, assume that there are vertices in Q' which are not equivalent. Then there is an edge a in Q' whose ends A, B are not equivalent. Let b be another edge adjacent to the vertex B and whose other vertex is C. Join A to C by a diagonal d. Then the edge b' to which the edge b must be glued is outside the triangle ABC. Otherwise, either $b = a$ or $b = a^{-1}$ which is contrary to the assumption that the vertices A and B are not equivalent or that there are no combinations of the form aa^{-1}. Now, apply the operation of subdivision along the diagonal d, and then the operation of gluing along the edge b (we glue it to b'). In the resultant development P', the set of vertices which are equivalent to A increases to one more in number, and the set of vertices equivalent to B reduces to one less (Fig. 43). In addition, if combinations of the form aa^{-1} appeared in the word of the development P', then we delete them by convolution. Moreover, it should be noted that the last reconstruction cannot alter the difference between the set of vertices, which are equivalent to B, and the set of those equivalent to A (verify this by yourself!).

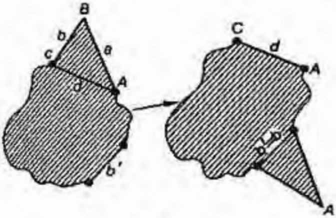


Fig. 43

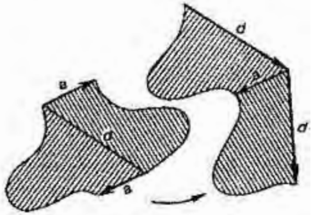


Fig. 44

Furthermore, if still some vertices which are not equivalent to A remain, then we repeat the whole process described until we obtain a development with the required property.

Thus, we shall assume from now on that all the vertices in our development are equivalent, and that there are no combinations of the form aa^{-1} in it.

(2) We show now that two similar letters in the word of a development can be always placed together. In fact, let two letters a and a be not placed together. Then draw the diagonal d joining the initial points of the two edges a and a in the polygon. By subdividing along d and then gluing along a, we find that there is no letter a in the new word as can readily be seen. However, the combination dd does appear, which is what we were striving for (Fig. 44). (It is easy to verify that the consequences of the first step are preserved.)

We perform the same procedure for other identical letters not placed together.

Note, moreover, that while applying the indicated procedure, we do not separate other combinations of the form aa, since only those edges which are adjacent to a are separated and they certainly are not equivalent to it.

(3) Assuming that the conditions of steps (1) and (2) are fulfilled, we will show that if the letters a and a^{-1} are not placed together in a word, then there are other letters b, b^{-1} such that the pairs a, a^{-1} and b, b^{-1} separate each other (Fig. 45).

We shall do this by employing reductio ad absurdum. If there is not a pair b, b^{-1} then there are only combinations of the form cc between a and a^{-1}. But this is

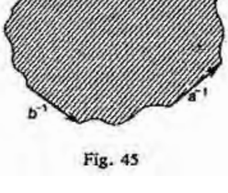


Fig. 45

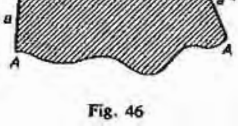


Fig. 46

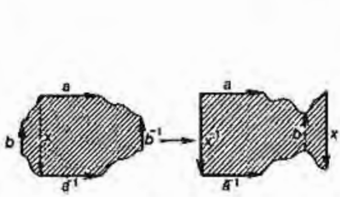


Fig. 47

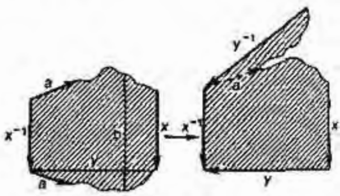


Fig. 48

contrary to the equivalence of all vertices of the development, since such a situation is possible only if the vertices A, B of the edge a are not equivalent (Fig. 46).

(4) Thus, there are two pairs in our word, a, a^{-1} and b, b^{-1}, that separate each other. We now demonstrate that these four can be always replaced by combinations of the form $xyx^{-1}y^{-1}$ whilst keeping the conditions of steps (1) and (2). First, join the origins of the edges a and a^{-1} with the diagonal x and make a subdivision along it; then glue along the edge b (Fig. 47). Join the ends of the edges x and x^{-1} in the polygon obtained by the diagonal y, subdivide along y again and then glue along a (Fig. 48).

We obtain a development in whose word there is the combination $xyx^{-1}y^{-1}$ instead of the letters a, b, a^{-1}, b^{-1}. If combinations of the form cc^{-1} appear after these operations, then they are removed by convolution while combinations of the form dd and $cdc^{-1}d^{-1}$ are not separated. Thus, the situation reached after steps (1) and (2) is preserved.

By applying the constructions of steps (1)-(4), we have transformed the original word to that consisting of combinations of the form $xyx^{-1}y^{-1}$ and aa. If there are no aa type combinations in the word, then this is a Type I canonical development.

(5) If there are both $xyx^{-1}y^{-1}$ and aa-type combinations then the word can be reduced to the Type II canonical form in the following way. Join the common vertex of the edges a and a to the common vertex of the edges y and x^{-1} with a diagonal d. Subdivide along d and glue along a (Fig. 49). The two pairs of the

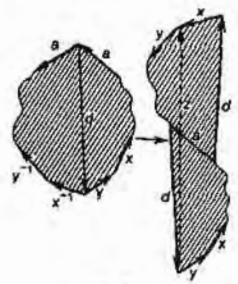


Fig. 49

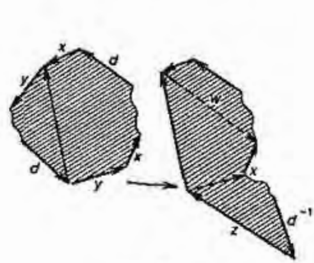


Fig. 50

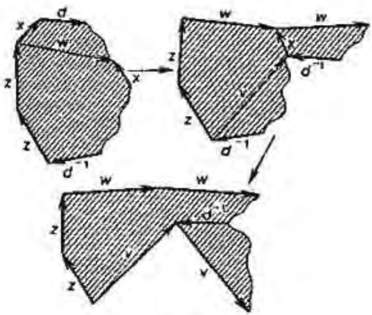


Fig. 51

separated edges thus obtained, viz., x and x, y and y, are then turned into the combinations zz, ww by applying step (2) (Figs. 50 and 51). After these operations are over, a separated pair d^{-1}, d^{-1} appears, which we turn into the combination vv (Fig. 51) again by performing step (2). We thus obtain a word of the required canonical form.

Hence, the pair of combinations $xyx^{-1}y^{-1}$, aa is replaced in the word by an arrangement of three aa-type pairs. In doing so, other $xyx^{-1}y^{-1}$ or aa combinations are not disturbed. The process can be repeated until all the $xyx^{-1}y^{-1}$ combinations have disappeared. ■

Exercise 2°. Verify that two closed surfaces X, X' whose developments are equivalent to canonical of the same type and with the same number m are homeomorphic.

4. The Euler Characteristic and Topological Classification of Surfaces. Let us now turn to the geometric interpretation of the theorem that we have just proved. It was shown in Sec. 3, Ch. I, that $xyx^{-1}y^{-1}$ combinations in the word of the canonical development of a surface X correspond to a handle, and an aa-form combination to a Möbius strip, both of these being glued to the remaining part of the surface X along its own boundary. Thus, if the canonical development of a surface is of Type I or II then this surface is glued together from a finite number of handles or a finite number of Möbius strips, respectively. This gluing can be easily represented as the result of gluing the handles or Möbius strips to the sphere S^2.

Consequently, we see that a surface with a Type I canonical development is an orientable M_p-type surface, where p is the number of the handles glued to the sphere (the *genus of the surface*). If, however, the canonical development of a surface is of Type II, then it is a non-orientable surface of the N_q type, $q \geqslant 1$, where q is the number of Möbius strips glued to the sphere (also the *genus of the surface*).

In the proof of the theorem, we have shown that if p handles and $q \geqslant 1$ Möbius strips are glued to the sphere, then the obtained surface is non-orientable and of the N_{2p+q} type.

The development classification theorem leads us to the conclusion that any closed surface is homeomorphic to a certain surface of the type M_p, N_q. To make the result more precise, consider the Euler characteristic of our surface. Let the decomposition of a surface X contain α_0 vertices, α_1 edges and α_2 images of polygons. The number $\chi(X) = \alpha_0 - \alpha_1 + \alpha_2$ is called the *Euler characteristic* of the surface. Obviously, this definition generalizes the one given earlier (see Sec. 3, Ch. I), since now the image of a polygon need not be, say, a topological polygon (the sides of a polygon may be glued together).

If X is of the M_p-type, and P its canonical development whose word is $a_1b_1a_1^{-1}b_1^{-1} \ldots a_pb_pa_p^{-1}b_p^{-1}$, then it is evident that $\alpha_0 = 1$, $\alpha_1 = 2p$, $\alpha_2 = 1$ and $\chi(X) = 2 - 2p$.

If X is of the N_q-type and $a_1a_1a_2a_2 \ldots a_qa_q$ is the word of its canonical development, then $\alpha_0 = 1$, $\alpha_1 = q$, $\alpha_2 = 1$ and $\chi(X) = 2 - q$.

If Q is an arbitrary development of the surface X then it can be transformed into a canonical development using elementary operations. It is easy to see that the elementary operations do not alter $\chi(X)$. In fact, a subdivision increases the numbers α_1 and α_2 by one, α_0 being unaltered; gluing decreases α_1 and α_2 by one, α_0 remaining unchanged, and a convolution decreases α_0 and α_1 by one. Therefore, the sum $\alpha_0 - \alpha_1 + \alpha_2$ remains unaltered. Hence, we can make the important conclusion that the canonical development P does not depend on the choice of elementary transformations of the development Q. In fact, if Q could be reduced to two canonical developments P, P', and both were, say, of Type I, i.e.,

$$a_1b_1a_1^{-1}b_1^{-1} \ldots a_pb_pa_p^{-1}b_p^{-1} \text{ and } \bar{a}_1\bar{b}_1\bar{a}_1^{-1}\bar{b}_1^{-1} \ldots \bar{a}_{p_1}\bar{b}_{p_1}\bar{a}_{p_1}^{-1}\bar{b}_{p_1}^{-1},$$

then the Euler characteristic calculated for a decomposition of Q would be the same as that evaluated for decompositions of P and P'. Hence, we would have the equality $2 - 2p = 2 - 2p_1$, whence $p = p_1$, i.e., the words for P and P' would coincide. Similar reasoning is held when the developments P and P' are both of Type II.

If, however, P is a development of Type I, and P' is of Type II, then the equality $2 - 2p = 2 - q$ is possible only if $q = 2p$. Therefore, the above reasoning merely establishes that a development cannot have two canonical forms, one of Type I and the other of Type II with the genera p and $q \neq 2p$. The general conclusion that the simultaneous reduction to both Types I and II canonical forms is impossible follows from the property to preserve its orientability (resp. non-orientability) of a development under elementary transformations (verify!).

Consequently, we have proved the first part of the following central theorem on the topological classification of surfaces.

THEOREM 2. *Any closed surface is topologically equivalent to a surface of the M_p- or N_q-type. Surfaces of the M_p- and N_q-, $q \geqslant 1$, types are not topologically equivalent if p and q are not both equal to zero; the surfaces M_p (or N_q) for different values of p (resp. q) are not topologically equivalent either*

The second part of the theorem was explained in Sec. 3, Ch. 1 (Item 4) and above. This explanation could have been considered to be the proof, had the topological invariance of the Euler characteristic $\chi(X)$ for an arbitrary closed surface X (we have only done so for $X = S^2$), and the fact that M_p and N_q are not homeomorphic, when $q = 2p, p > 0$, been proved. These facts will be established in Sec. 4, Ch. III, using the idea of the fundamental group of a space.

Exercises.

3°. Draw the diagram for gluing a surface whose canonical development has the word

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}a_3b_3a_3^{-1}b_3^{-1}.$$

4°. Draw the diagram for gluing a surface characterized by the word $a_1a_1a_2a_2a_3a_3$. Indicate the type and genus of this surface.

5°. Verify that the following closed surfaces have the indicated type and genus:
(1) the sphere has $M_0 = N_0$;
(2) the torus (i.e., the sphere with one handle) M_1;
(3) the double torus (i.e., the sphere with two handles) M_2;
(4) the projective plane N_1;
(5) the Klein bottle N_2.
Draw the diagrams of their decompositions.

6°. Call a topological space in which each point has a neighbourhood homeomorphic to an open interval of the number line, a *one-dimensional manifold* M^1. Call the decomposition of M^1 into *arcs* which are the topological images of the line-segment $[0, 1]$ and whose ends are adjacent to each other (i.e., meet at vertices) a *triangulation* of M^1; we assume that M^1 consists of a finite number of arcs.
Prove that a triangulable manifold M^1 is homeomorphic to the circumference S^1 or some of its replicas.

5. ORBIT SPACES. PROJECTIVE AND LENS SPACES

1. The Definition of an Orbit Space. We consider here important examples of factor spaces arising when groups act on topological spaces.

Let $H(X)$ be the set of all homeomorphisms of a topological space X onto itself. The product of two homeomorphisms h_1 and h_2 is defined as follows: $(h_1h_2)(x) = h_2(h_1(x))$. In addition, for each $h \in H(X)$, there is an inverse mapping $h^{-1} \in H(X)$ and $hh^{-1} = h^{-1}h = 1_X$. Thus, $H(X)$ is a multiplicative group (non-commutative, generally speaking) with the identity element 1_X.

DEFINITION 1. We will say that an abstract group G *acts* (*from the left*) *on a space* X if a homeomorphism of the group G into group $H(X)$ is given.

If G acts on X then to each $g \in G$, there corresponds $h_g \in H(X)$:

$$g \to h_g;\ g_1g_2 \to h_{g_1}h_{g_2},\ g^{-1} \to (h_g)^{-1},\ 1_G \to 1_X.$$

Let $x \in X$ be an arbitrary point; the set $\bigcup_{g \in G} h_g(x)$ is called its *orbit* and denoted by O_x.

Exercise 1°. Show that two orbits O_x, O_y either coincide or are disjoint.

The last statement enables us to introduce on X an equivalence $R: x \overset{R}{\sim} y \Leftrightarrow$ $\Leftrightarrow O_x = O_y$, i.e., when x and y belong to the same orbit.

DEFINITION 2. The factor space X/R is called the *orbit space of the group* G and denoted by X/G.

This method of constructing factor spaces plays an important role in modern topology. Consider some examples.

2. **Projective Spaces RP^n, CP^n.** Consider the sphere $S^n \subset R^{n+1}$. Let each point $x = (\xi_1, \ldots, \xi_{n+1}) \in S^n$ be associated with its diametrically opposite point $Ax = (-\xi_1, \ldots, -\xi_{n+1}) \in S^n$. The mapping $A: S^n \to S^n$ is a homeomorphism and called a *central symmetry*. The following relations are obvious: $A = A^{-1}$, $A^2 = 1_{S^n}$. Therefore, the set $\{A, 1_{S^n}\}$ is a group (multiplicative) consisting of two elements; it is isomorphic to the group Z_2 (additive) of residues mod 2. Consequently, the action of Z_2 on S^n is defined.

DEFINITION 3. The space S^n/Z_2 is called the *real projective space* and denoted by RP^n.

Thus, RP^n can be obtained from S^n by identifying diametrically opposite points $x, -x$.

Consider the set $G = R \setminus 0$ (i.e., all real numbers except zero). This is a multiplicative group and we define its action on the space $X = R^{n+1} \setminus \{0\}$ to be

$$h_\lambda(x) = \lambda x,\ \lambda \in G,\ x \in R^{n+1} \setminus \{0\}.$$

O_x is obviously the set of all the points of the straight line in R^{n+1}, passing through O and x, without the point 0. Therefore, $(R^{n+1} \setminus \{0\})/G$ is the set of all straight lines in R^{n+1} passing through the origin. The space $(R^{n+1} \setminus \{0\})/G$ is homeomorphic to RP^n. The homeomorphism is established by the correspondence in which a pair $(x, -x)$ is associated with the straight line passing through the points x and $-x$.

Exercises.

2°. Describe the topology of $(R^{n+1} \setminus \{0\})/G$ and verify that the spaces RP^n and $(R^{n+1} \setminus \{0\})/G$ are homeomorphic.

3°. Glue the diametrically opposite points of the boundary of the disc $\bar{D}^n$ together. Show that the factor space obtained is homeomorphic to RP^n.

Consider now the complex space C^{n+1}. Let $G = C \setminus \{0\}$ be the multiplicative group of complex numbers. It acts in $C^{n+1} \setminus \{0\}$ by the rule $h_\lambda x = \lambda x$, $\lambda \in G$, $x \in C^{n+1} \setminus \{0\}$. Therefore, $(C^{n+1} \setminus \{0\})/G$ can be identified with the set of all complex straight lines in C^{n+1} passing through zero.

DEFINITION 4. The space $(C^{n+1} \setminus \{0\})/G$ is termed the *complex projective space* and denoted by CP^n.

We shall construct another model of CP^n. Consider the unit sphere in C^{n+1}, viz., $S^n_C = \{x: |\xi_1|^2 + \ldots + |\xi_{n+1}|^2 = 1\}$. The group $G = \{e^{i\alpha}, 0 \leqslant \alpha < 2\pi\}$

acts on it according to the rule $e^{i\alpha}x = (e^{i\alpha}\xi_1, e^{i\alpha}\xi_2, \ldots, e^{i\alpha}\xi_{n+1})$. Thus, group G can be identified with the unit circumference S^1 in the complex plane C. Hence, S^1 acts on the coordinate $\xi_i \in C$, and the orbit of the point ξ_i in C is the circumference of radius $|\xi_i|$ if $|\xi_i| \neq 0$. Therefore, the orbit $O_x = \{e^{i\alpha}x\}$ $(0 \leqslant \alpha < 2\pi)$ of each point $x \in S_C^n$ is a great circle on S_C^n. But S_C^n can be identified with S^{2n+1} and O_x can be assumed to be a great circle on S^{2n+1}; therefore, the action $G = S^1$ has been determined on S^{2n+1} and we have the homeomorphisms

$$S_C^n/S^1 \approx S^{2n+1}/S^1 \approx CP^n.$$

The first homeomorphism has already been constructed. We shall now establish the homeomorphism $S_C^n/S^1 \approx (C^{n+1} \setminus \{0\})/G$. We define this homeomorphism by assigning to each complex straight line (i.e., a point of CP^n) that great circle on S_C^n (a point of S_C^n/S^1), which is the intersection of the complex straight line with S_C^n

3. Lens Spaces. At the end of Item 2, we dealt with the group S^1 of complex numbers whose moduli were equal to unity, and which acted in the complex plane C.

Consider those finite subgroups of the group S^1 which are known to be finite cyclic and isomorphic to the additive groups Z_k of residues mod k. Let such a group Z_k act in the j-th replica of the space C as follows:

$$\xi_j \to e^{2\pi i \frac{k_{j-1}}{k}} \xi_j,$$

where k_j is a certain integer, $0 \leqslant k_j < k$. Then the action of Z_k is determined in C^{n+1} and in S_C^n thus:

$$(\xi_1, \xi_2, \ldots, \xi_j, \ldots, \xi_{n+1})$$
$$\to \left(e^{2\pi i \frac{1}{k}}\xi_1, e^{2\pi i \frac{k_1}{k}}\xi_2, \ldots, e^{2\pi i \frac{k_j}{k}}\xi_{j+1}, \ldots, e^{2\pi i \frac{k_n}{k}}\xi_{n+1}\right).$$

DEFINITION 5. The space S_C^n/Z_k is called a *generalized lens space* if any k_i and k are relatively prime, and denoted by $L(k, k_1, \ldots, k_n)$. When $n = 1$, the space $L(k, k_1)$ is called a *lens space*.

Exercises.

4°. Show that if any k_j and k are relatively prime, then each orbit of the action of the group Z_k described above consists of k points.

5°. Show that the following formula determines the action of the group S^1 on a generalized lens space:

$$e^{i\alpha}(\xi_1, \ldots, \xi_{n+1}) = \left(e^{i\frac{\alpha}{k}}\xi_1, \ldots, e^{i\frac{\alpha}{k}}\xi_{n+1}\right).$$

6°. Show that $L(k, k_1, \ldots, k_n)/S^1 = CP^n$.

6. OPERATIONS OVER SETS IN A TOPOLOGICAL SPACE

In this section, we again turn our attention to the investigation of the properties of topological spaces and consider the closure and interior operations and the boundary operators on a set and also two notions closely related to them, that is, the concepts of limit and boundary points. All these ideas generalize well-known concepts of mathematical analysis.

1. The Closure of a Set.

Let (X, τ) be a topological space.

DEFINITION 1. We define the closure $\bar{A}$ of a set $A \subset X$ to be the intersection of all closed sets containing A.

The following statements are obvious:

(1) The closure $\bar{A}$ is the smallest closed set containing A.

(2) If A is closed then $\bar{A} = A$.

A closed set can be characterized by limit points, which we do below.

DEFINITION 2. A point $x \in X$ is said to be a *limit* one of a given set $A \subset X$ if there is at least one point $x' \in A$ other than x in each neighbourhood $\Omega(x)$ of the point x.

Exercise 1°. Verify that this definition can be restricted to only open neighbourhoods of the point x.

EXAMPLES. Consider the sets $A = \{n\}$, $B = \left\{\frac{1}{n}\right\}$, $n = 1, 2, \ldots$, $C = (0, 1)$, $D = [0, 1]$ in R^1. The set A has no limit points, the set B has one limit point, i.e., 0, and the limit points of the sets C and D fill the whole line-segment $[0, 1]$.

The notion of limit point in a topological space is, as can easily be seen, a generalization of the concept of limit point in analysis. We shall now prove some useful statements associated with limit points.

THEOREM 1. *A set $A \subset X$ is closed if and only if it contains all its limit points.*

PROOF. Let A be closed, x a limit point of A, and $x \overline{\in} A$. Hence, $x \in X \setminus A = \Omega(x)$ which is an open set, a neighbourhood of the point x. Therefore, $\Omega(x) \cap A = \varnothing$, which is contrary to the definition of a limit point.

Let A contain all its limit points. To show that it is closed, i.e., that its complement $U = X \setminus A$ is open, it suffices to prove that for any point $x \in U$, there is a neighbourhood $\Omega(x)$ such that $\Omega(x) \subset U$. By assuming the contrary, we find that for a certain point $x_0 \in U$ and any its neighbourhood $\Omega(x_0)$, there is a point $x' \in \Omega(x_0)$ such that $x' \overline{\in} U$. Then $x' \in X \setminus U = A$, therefore, x_0 is a limit point of A, which is contrary to $x_0 \overline{\in} A$. ■

The set A' of all limit points of a set A is called the *derived set* of A. Thus, a new operation arises, viz., assigning to each set $A \subset X$ the derived set A'.

THEOREM 2. *For any set $A \subset X$, the set $A \cup A'$ is closed.*

PROOF. We will show that the set $X \setminus (A \cup A')$ is open. Let x be an arbitrary point of $X \setminus (A \cup A')$. Then x is not a limit point of A, therefore, there is its neighbourhood $\Omega(x)$ such that $\Omega(x) \cap A = \varnothing$. Let $x' \in \Omega(x)$ be an arbitrary point.

Then for any neighbourhood $V(x')$ of the point x' such that $V(x') \subset \Omega(x)$, we have $V(x') \cap A = \varnothing$, therefore, x' is not a limit point of A and $\Omega(x) \cap A' = \varnothing$. Thus, $\Omega(x) \subset X \setminus (A \cup A')$, and because x is arbitrary, the set $X \setminus (A \cup A')$ is open. ■

Exercise 2°. (1) Verify that $(A \cup B)' = A' \cup B'$, $(A \cap B)' \subset A' \cap B'$ and $(A \setminus B)' \supset A' \setminus B'$.
(2) Let $X = \{a, b\}$ be a space of two elements equipped with the topology consisting of the three sets: $\varnothing$, X, $\{a\}$. Give an example of a set $A \subset X$ for which the inclusion $(A')' \subset A'$ is not valid.

We shall now prove a basic statement about the structure of the closure of a set.

THEOREM 3. $\overline{A} = A \cup A'$ *for any set* $A \subset X$.

PROOF. By Theorem 2, the set $A \cup A'$ is closed. Therefore, by the definition of a closure, $\overline{A} \subset A \cup A'$. On the other hand, it is obvious that any closed set containing A also contains all limit points of A, and therefore contains A'. Hence, $A \cup A' \subset \overline{A}$. Thus, $\overline{A} = A \cup A'$.

Exercise 3°. Let *A be the set of rational points on the real straight line* R^1. Show that $\overline{A} = R^1$.

If a topological space X has a countable subset A whose closure coincides with X, then it is said to be *separable*. It is easy to verify that separability is a topological property.

Exercises.
4°. Show that the space R^n, the disc D^n, and the sphere S^{n-1} are separable.
5°. Verify the following properties of the closure operation: $\overline{A \cup B} = \overline{A} \cup \overline{B}$, $\overline{\overline{A}} = \overline{A}$, $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$, $\overline{A} \setminus \overline{B} \subset \overline{A \setminus B}$.
6°. Let Y be a subspace of a topological space X and A a subset of Y. Denote the closure of the set A in the subspace Y by $\overline{A}_Y$, and the closure of A in X by $\overline{A}$. Show that $\overline{A}_Y = \overline{A} \cap Y$.

DEFINITION 3. A point $x \in A$ is said to be *isolated* if there is a neighbourhood $\Omega(x)$ of the point x such that it does not contain any points of the set A other than x.

A point $x \in A$ is isolated if and only if $x \in A \setminus A'$.

DEFINITION 4. A set A is said to be *discrete* if each of its points is isolated.

2. The Interior of a Set.

Consider two other important notions connected with that of neighbourhood.

DEFINITION 5. A point $x \in A$ is called an *interior point* of a set A if it has a neighbourhood $\Omega(x)$ such that $\Omega(x) \subset A$.

The set of all interior points of a set A is called the *interior* of A and denoted by Int A.

EXAMPLE. Consider $A = [0, 1]$, the line-segment of the real straight line R^1. It is easy to see that Int $[0, 1] = (0, 1)$.

The operation Int is dual of the closure operation, which follows from its properties as enunciated in the following theorem.

THEOREM 4. *For any set $A \subset X$, we have*: (1) $\operatorname{Int} A$ *is an open set*; (2) $\operatorname{Int} A$ *is the largest open set contained in* A; (3) (A *is open*) $\Leftrightarrow$ ($\operatorname{Int} A = A$); (4) ($x \in \operatorname{Int} A$) $\Leftrightarrow$ ($x \in A$ *and x is not a limit point of* $X \setminus A$); (5) $\overline{X \setminus A} = X \setminus \operatorname{Int} A$.

PROOF. Properties (1)-(3) are almost evident. We will verify, for example, Property (1). Let $x \in \operatorname{Int} A$. Then there is an open neighbourhood $U(x)$ of the point x such that $U(x) \subset A$. Therefore, $\operatorname{Int} A$ is a neighbourhood of each of its points and hence an open set.

As for property (4), if $x \in \operatorname{Int} A$ then, obviously, $x \in A$ and $x \notin (X \setminus A)'$. Conversely, if $x \in A$ and $x \notin (X \setminus A)'$ then there is a neighbourhood $\Omega(x) \subset A$, therefore, $x \in \operatorname{Int} A$.

The verification of property (5) is left to the reader. ■

The set $\operatorname{Int}(X \setminus A)$ has to be considered quite often. It is called the *exterior* of the set A and denoted by $\operatorname{ext} A$.

Exercise 7°. Show that $\bar{A} = X \setminus \operatorname{ext} A$.

3. The Boundary of a Set.

The following important concepts are those of boundary point and the boundary of a set A. They are associated with the intuitive idea of a 'Separator' between a region of Euclidean space and its exterior. Some examples will be considered in the next section. Here is the general definition.

DEFINITION 6. We call the set $X \setminus (\operatorname{Int} A \cup \operatorname{ext} A)$ the *boundary* ∂A of a set A, and call every point of the boundary a *boundary point* of the set A.

Thus, $x \in \partial A$ if and only if each neighbourhood of x contains a point from both A and $X \setminus A$.

EXAMPLE. Let $X = R^1$ and $A = (0, 1]$. Then $\operatorname{Int} A = (0, 1)$, $X \setminus A = (-\infty, 0] \cup (1, +\infty)$, $\operatorname{Int}(X \setminus A) = (-\infty, 0) \cup (1, +\infty)$. Therefore, $\partial A = \{0, 1\}$ is the set of two points: 0 and 1.

Thus, we have the *boundary operator* ∂. Its relation to the closure and Int operations is cleared up by the following theorem.

THEOREM 5. *For any $A \subset X$, we have*: (1) $\partial A = \bar{A} \cap \overline{(X \setminus A)}$; (2) $\partial A = \bar{A} \setminus \operatorname{Int} A$; (3) $\bar{A} = A \cup \partial A$; (4) $\operatorname{Int} A = A \setminus \partial A$; (5) ($A$ *is closed*) $\Leftrightarrow$ ($\partial A \subset A$); (6) (A *is open*) $\Leftrightarrow$ ($(\partial A) \cap A = \varnothing$).

PROOF. We shall prove some of these statements and leave the others as exercises.

(1) Let $x \in \partial A$, then in any neighbourhood $U(x)$ of the point x, there are points x_1, x_2 such that $x_1 \in A$, and $x_2 \in X \setminus A$. Hence $x \in \bar{A}$ and $x \in \overline{X \setminus A}$, i.e., $x \in \bar{A} \cap \overline{(X \setminus A)}$. Conversely, if $x \in \bar{A} \cap \overline{(X \setminus A)}$, then $x \in \bar{A}$, $x \in \overline{(X \setminus A)}$. Since $\overline{(X \setminus A)} = X \setminus \operatorname{Int} A$, $\bar{A} = X \setminus \operatorname{ext} A$ (see Item 5 of Theorem 4 and Exercise 7), we find that $x \notin \operatorname{Int} A$, $x \notin \operatorname{ext} A$, whence $x \in \partial A$.

(2) By definition, $\partial A = X \setminus (\operatorname{Int} A \cup \operatorname{ext} A) = (X \setminus \operatorname{ext} A) \setminus \operatorname{Int} A = \bar{A} \setminus \operatorname{Int} A$.

(3) Since $\operatorname{Int} A \subset \bar{A}$, it follows from (2) that $\bar{A} = \operatorname{Int} A \cup \partial A \subset A \cup \partial A$; since $\partial A \subset \bar{A}$, we see that $A \cup \partial A \subset A \cup \bar{A} = \bar{A}$.

(5) If A is closed then $\partial A \subset \bar{A} = A$. Conversely, if $\partial A \subset A$ then by (3), $\bar{A} = A \cup \partial A$ (see Item 3) whence $A = \bar{A}$, i.e., A is closed. ■

Exercises.
8°. Let U be open in X and $A = \partial U$. Show that $\partial A = A$. Prove the converse statement.
9°. Let Y be a subspace of a topological space X, and A a subset in Y. Denote the boundary of the set A in Y by $\partial_Y A$, and the boundary of A in X by ∂A. Verify that it is not always true that $\partial_Y A = (\partial A) \cap Y$. Give some examples.

7. OPERATIONS OVER SETS IN METRIC SPACES. SPHERES AND BALLS. COMPLETENESS

1. Operations Over Sets in Metric Spaces. Here we consider the concepts studied in the previous section as they apply to metric spaces. Remember that the base for the topology in (M, ρ) consists of all possible balls $D_r(x_0)$, where $r > 0$ is the radius, and x_0 is the centre of the ball. The metric ρ makes it possible for us to speak of convergent sequences in M (see Sec. 2, Ch. 1). We can express $\bar{A}$, A', Int A, ∂A in these terms thus:

(a) the condition $x \in \text{Int}\, A$ is equivalent to the ball $D_\varepsilon(x)$, for a certain $\varepsilon > 0$, being contained wholly in A; this follows from the definition of the metric topology τ_ρ;

(b) the condition $x \in A'$ is equivalent to the existence of a sequence $\{a_n\}$ convergent to x, where $a_n \in A$, $a_n \neq x$.

In fact, if $x \in A'$ then for any $r_1 > 0$, there is an element a_1 in A such that $a_1 \in D_{r_1}(x)$, $a_1 \neq x$. Let $0 < r_2 < \rho(x, a_1)$, then again there is an element $a_2 \in D_{r_2}(x)$, $a_2 \neq x$, etc. Thus, the sequences $\{r_n\}$ and $\{a_n\} \subset A$ are constructed such that $\rho(\bar{a}_n, x) < r_n$, $r_n \to 0$, $a_n \neq x$, i.e., $a_n \to x$.

Conversely, let there exist a sequence $a_n \to x$, where $a_n \neq x$, $a_n \in A$. Then for any neighbourhood $\Omega(x)$ of the point x, there exist a ball $D_\varepsilon(x) \subset \Omega(x)$ and $N(\varepsilon)$ such that $\rho(a_n, x) < \varepsilon$ for $n \geqslant N(\varepsilon)$. Hence $a_n \in \Omega(x)$ when $n \geqslant N(\varepsilon)$ and $a_n \neq x$, which completes the proof.

The definition of a limit point in terms of sequences convergent to it given above is always used in analysis as the definition of a limit point of a set;

(c) the condition that a set A is closed implies, just like for a topological space, that A contains all its limit points. This condition is equivalent to the fact that the condition $x \in A$ follows from the existence of a sequence $\{a_n\} \subset A$ convergent to x. In fact, the condition that A is closed is equivalent, for example, to the condition that $A' \subset A$ (see Sec. 5) which is equivalent to the previous statement;

(d) the condition $x \in \partial A$ is equivalent to $D_r(x) \cap A \neq \varnothing$ and $D_r(x) \cap (X \setminus A) \neq \varnothing$ for any $r > 0$, i.e., any ball with centre at the point x will 'scoop' out the points of A and $X \setminus A$. This statement is obvious.

We are also giving an equivalent definition which is often used in analysis;

(e) the condition $x \in \partial A$ is equivalent to the existence of a sequence $\{a_n'\} \in X \setminus A$ convergent to x, and to the existence of a sequence $\{a_n\} \subset A$ convergent to x.

In fact, suppose $x \in \partial A$. Then for any $r > 0$, the ball $D_r(x)$ 'scoops' points out of both A (i.e., the point a_r) and $X \setminus A$ (i.e., the point a_r'). Assuming that $r = r_n$, $r_n \to 0$, we obtain the sequences $a_{r_n} \in A$, $a_{r_n}' \in X \setminus A$ such that $a_{r_n} \to x$, $a_{r_n}' \to x$.

Conversely, if $a_n \to x$, $\{a_n\} \subset A$ and $a_n' \to x$, $\{a_n'\} \subset X \setminus A$, then any ball $D_r(x)$ contains both the point a_n and the point a_n' for a sufficiently large $n = n(r)$; therefore, $x \in \partial A$.

2. **Balls and Spheres in R^n.** We shall investigate the sphere S^n, the open disc D^{n+1} and the closed disc $\bar{D}^{n+1}$ in R^{n+1}.

THEOREM 1. *The following equalities are valid*: $\bar{D}^{n+1} = (D^{n+1})' = \overline{(D^{n+1})}$.

PROOF. If the 'ray' $\{tx_0\}$, $0 \leqslant t < +\infty$, is considered (it emanates from the centre of the ball, the point O, and passes through the point $x_0 \in \bar{D}^{n+1}$, $x_0 \neq 0$), then the points $x_k = \dfrac{k-1}{k} x_0$ of this ray tend to x_0 and lie in D^{n+1} (verify this by using the metric on R^{n+1}), and the points $y_k = \dfrac{1}{k} x_0$ also lie in D^{n+1} and tend to zero. Therefore, $(D^{n+1})' \supset \bar{D}^{n+1}$. On the other hand, $\overline{(D^{n+1})} \subset \bar{D}^{n+1}$ (here $\overline{(D^{n+1})}$ is the topological closure of the ball D^{n+1}). In fact, if $x_k \to y$, $x_k \in D^{n+1}$, i.e., if $y \in (D^{n+1})'$ then

$$\rho(y, 0) \leqslant \rho(y, x_k) + \rho(x_k, 0) < \rho(y, x_k) + 1,$$

whence by taking into account that $\rho(y, x_k) \to 0$ as $k \to \infty$, we have $\rho(y, 0) \leqslant 1$, i.e., $y \in \bar{D}^{n+1}$.

After combining the inclusion relations that we have obtained with the evident relation $(D^{n+1})' \subset \overline{(D^{n+1})}$, we have

$$\bar{D}^{n+1} \subset (D^{n+1})' \subset \overline{(D^{n+1})} \subset \bar{D}^{n+1},$$

whence the statement of the theorem readily follows. ■

THEOREM 2. *The sphere is the boundary of a ball*: $S^n = \partial(D^{n+1})$.

PROOF. Let $x_0 \in S^n$ ($S^n \neq \varnothing$!). Then $x_k = \dfrac{k-1}{k} x_0 \in D^{n+1}$, and the sequence $\{x_k\}$ converges to x_0 as $k \to \infty$. Therefore, $S^n \subset \partial(D^{n+1})$. Conversely, let $x_0 \in \partial(D^{n+1})$. Then $x_0 \bar{\in} D^{n+1}$ since D^{n+1} consists of interior points, and there exists a sequence $\{x_k\} \in D^{n+1}$ convergent to x_0 (see Item 1, (e)). Therefore, $x_0 \in (D^{n+1})' = \bar{D}^{n+1}$, $x_0 \in S^n$. ■

Exercises.

1°. Prove that $S^n = \partial(\bar{D}^{n+1})$.

2°. Let $\varphi: R^n \to R^1$ be a continuous function. Prove that the set $A = \{x \in R^n : \varphi(x) < \alpha\}$ is open, and the sets $B = \{x \in R^n : \varphi(x) \leqslant \alpha\}$, $C = \{x \in R^n : \varphi(x) = \alpha\}$ are closed for any $\alpha \in R^1$ (these sets are called the *Lebesgue sets* of the function φ).

3°. Show for the sets of Exercise 2° that $\bar{A} \subset B$. Give an example when $\bar{A} = B$, $\partial A = C$, and also an example when $\bar{A} \neq B$ and $\partial A \neq C$.

3. **Balls and Spheres in Arbitrary Metric Spaces.** Consider a metric space (M, ρ). We define a *closed ball* $\bar{D}_r(x_0)$ and *sphere* $S_r(x_0)$ (i.e., with radius

$r > 0$ and centre at the point x_0) by the equalities

$$\bar{D}_r(x_0) = \{x \in M, \rho(x, x_0) \leqslant r\}, S_r(x_0) = \{x \in M, \rho(x, x_0) = r\}.$$

Note that $\bar{D}_r(x_0)$, $S_r(x_0)$ are closed sets in M. In fact, if $\{x_n\} \in \bar{D}_r(x_0)$ and $x_n \to y$ then

$$\rho(x_0, y) \leqslant \rho(x_0, x_n) + \rho(x_n, y) \leqslant r + \rho(x_n, y),$$

whence $\rho(x_0, y) \leqslant r$, i.e., $y \in \bar{D}_r(x_0)$; $S_r(x_0)$ is closed as the complement in the closed set $\bar{D}_r(x_0)$ to the open set $D_r(x_0)$.

Are the theorems of Item 2 valid in the metric space? The following example proves that the answer is negative.

EXAMPLE 1 (COUNTEREXAMPLE). Let M be a finite set. We specify a metric $\rho(x, x) = 0$, $\rho(x, y) = 1$ when $x \neq y$. Then for $r < 1$,

$$D_r(x_0) = \{x_0\}, \bar{D}_r(x_0) = \{x_0\}, S_r(x_0) = \varnothing$$

and

$$(\overline{D_r(x_0)}) = \bar{D}_r(x_0) \neq (D_r(x_0))' = \varnothing.$$

However,

$$S_r(x_0) = \partial D_r(x_0) = \varnothing.$$

When $r = 1$ $D_1(x) = \{x_0\}$, $\bar{D}_1(x_0) = M$, $S_1(x_0) = M \setminus \{x_0\}$ and $(\overline{D_1(x_0)}) \subset \overline{D_1}(x_0)$. Furthermore, $(\overline{D_1(x_0)}) \neq \bar{D}_1(x_0)$, $S_1(x_0) \neq \partial D_1(x_0) = \varnothing$.
Finally, when $r > 1$, we have

$$D_r(x_0) = \bar{D}_r(x_0) = M, S_r(x_0) = \varnothing;$$

moreover, $(\overline{D_r(x_0)}) = \bar{D}_r(x_0) \neq (D_r(x_0))^{\cdot} = \varnothing$, $S_r(x_0 = \partial D_r(x_0) = \varnothing$.

The following theorem provides a necessary and sufficient condition for a sphere in a metric space to be the boundary of a ball.

THEOREM 3. *The following equivalence is valid:*

$$(S_r(x_0) = \partial D_r(x_0)) \Leftrightarrow (\overline{D_r(x_0)} = \bar{D}_r(x_0)).$$

PROOF. It follows from $(\overline{D_r(x_0)}) = \bar{D}_r(x_0)$ that

$$S_r(x_0) = \bar{D}_r(x_0) \setminus D_r(x_0) = \overline{D_r(x_0)} \setminus D_r(x_0) = \partial D_r(x_0).$$

Conversely, if $S_r(x_0) = \partial D_r(x_0)$, then

$$\overline{D_r(x_0)} = D_r(x_0) \cup \partial D_r(x_0) = D_r(x_0) \cup S_r(x_0) = \bar{D}_r(x_0). \blacksquare$$

Exercise 4°. Let $M = C_{[0,1]}$ be the space of continuous functions with the standard metric (see Sec. 2, Ch. I). Give one interpretation of D_r, $\bar{D}_r$, S_r and show that $S_r = \partial D_r$

4. Completeness of Metric Spaces. In analysis, Cauchy's criterion of the convergence of a sequence of numbers (in the space R^1) has been established: a sequence $\{x_n\}$ converges to some point $x_0 (x_n \to x_0)$ if and only if it is *fundamental*,

i.e., for any $\varepsilon > 0$, there is an integer $N(\varepsilon)$ such that $|x_{n+m} - x_n| < \varepsilon$ as soon as $n \geqslant N(\varepsilon)$, $m \geqslant 1$.

If $x_n \xrightarrow{\rho} x_0$ in (M, ρ), then $\{x_n\}$ can be easily shown to be fundamental as is true in the case of R^1, i.e., for any $\varepsilon > 0$, there is $N(\varepsilon)$ such that

$$\rho(x_{n+m}, x_n) \leqslant \varepsilon, n \geqslant N(\varepsilon), m \geqslant 1. \qquad (1)$$

However, the converse is not always true.

DEFINITION 1. The space (M, ρ) in which Cauchy's criterion holds true (i.e., any fundamental sequence has a limit) is called a *complete space*.

EXAMPLES

2. Let $M = Q \subset R^1$ be the set of rational numbers in R^1. This metric space is not complete since there exist sequences of rational numbers convergent to an irrational number (i.e., fundamental, but having no limit in Q).
3. The space $M = R^1$ is complete.
4. The space $M = R^n$ is complete. This follows from the fact that the fundamentality or convergence of a sequence of lines $\{(\xi_1^k, \ldots, \xi_n^k)\}$ of length n is equivalent to the fact that sequences of the numbers $\{\xi_i^k\}$, $i = 1, \ldots, n$, are fundamental or convergent.

Exercise 5°. Prove that the space $M = C^n$ is complete.

EXAMPLE 5. The space $M = C_{[0, 1]}$ is complete in the metric

$$\rho_1(x(t), y(t)) = \max_{0 \leqslant t \leqslant 1} |x(t) - y(t)|$$

but not in the metric

$$\rho_2(x(t), y(t)) = \left\{ \int_0^1 |x(t) - y(t)|^2 dt \right\}^{1/2}. \qquad (2)$$

The statement just formulated is proved in analysis.

The examples show that the property of completeness is not topological, i.e., generally speaking, it is not preserved under homeomorphisms (e.g., the interval $(a, b) \subset R^1$ and R^1 itself are homeomorphic but (a, b) is not complete).

The following statement holds true but we leave its proof to the reader as an exercise.

THEOREM 4. *Let (M, ρ) be a metric space, and $M_1 \subset M$ a subspace. If M_1 is complete then it is closed in M; if M is complete and M_1 is closed in M, then M_1 is complete.*

8. PROPERTIES OF CONTINUOUS MAPPINGS

1. **Equivalent Definitions of a Continuous Mapping.** We shall express the property of the continuity of a mapping $f: X \to Y$ of topological spaces X and Y in terms of other topological concepts, i.e., of the neighbourhood and the closure of a set.

THEOREM 1. *Let $f: X \to Y$ be a continuous mapping. The following properties are equivalent:* (1) *f is continuous;* (2) *for any $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$;* (3) *for any $A \subset Y$, we have: $\overline{f^{-1}(A)} \subset f^{-1}(\bar{A})$.*

PROOF. We shall prove some implications. (1) $\Rightarrow$ (2): We conclude from the definition of continuity that the set $f^{-1}\overline{(f(A))}$ is closed in Y and contains A. Therefore, $\bar{A} \subset f^{-1}(\overline{f(A)})$, whence $f(\bar{A}) \subset \overline{f(A)}$. (2) $\Rightarrow$ (1): Obviously, it follows from (2) that $\bar{A} \subset f^{-1}(\overline{f(A)})$ for any A. Choosing $A = f^{-1}(F)$, where F is an arbitrary closed set in Y, we obtain that $\overline{f^{-1}(F)} \subset f^{-1}(\overline{f(f^{-1}(F))}) = f^{-1}(F)$. Therefore, $f^{-1}(F)$ is closed for any closed $F \subset Y$, i.e., f is continuous. (1) $\Rightarrow$ (3): The continuity of f implies the closedness of $f^{-1}(\bar{A})$. Hence, (3) immediately follows from the inclusion $f^{-1}(A) \subset f^{-1}(\bar{A})$. (3) $\Rightarrow$ (1): For a closed A, the chain of inclusion relations $f^{-1}(A) \supset \overline{f^{-1}(A)} \supset f^{-1}(A)$ follows from (3). Whence $f^{-1}(A)$ is closed, and the mapping f is therefore continuous. ■

On the analogy of the definition of the continuity of a mapping in a metric space, continuous mappings of topological spaces can be defined as being continuous at every point by introducing the notion of continuity of a mapping at a point in a topological space.

DEFINITION. A mapping $f : X \to Y$ of topological spaces is *continuous at a point* $x_0 \in X$ if for any neighbourhood $\Omega(f(x_0))$ of the point $f(x_0)$, there exists a neighbourhood $\Omega(x_0)$ of the point x_0 such that $f(\Omega(x_0)) \subset \Omega(f(x_0))$.

Exercise 1°. The following property of a mapping $f : X \to Y$ is equivalent to the continuity at a point: the full inverse image $f^{-1}(\Omega(f(x_0)))$ of any neighbourhood of the point $f(x_0)$ is a neighbourhood of the point x_0.

THEOREM 2. *A mapping $f : X \to Y$ is continuous if and only if it is continuous at each point $x \in X$.*

PROOF. Let $f : X \to Y$ be continuous, $x_0 \in X$ an arbitrary point, and $\Omega(f(x_0))$ an arbitrary neighbourhood of the point $f(x_0)$. Then there is an open set $V \subset Y$ such that $V \subset \Omega(f(x_0))$ and $f(x_0) \in V$. Put $U = f^{-1}(V)$. U is an open set and $x_0 \in U$. Then $f(U) \subset \Omega(f(x_0))$, which proves the continuity of f at the point x_0.

Conversely, let f be continuous at every point $x \in X$. Let $V \subset Y$ be an arbitrary open set and let $A = f^{-1}(V)$. Since V is a neighbourhood of each of its points and f is continuous at each of its points, for any $x \in A$, there is an $\Omega(x)$-neighbourhood of the point x such that $f(\Omega(x)) \subset V$. Therefore, $\Omega(x) \subset A$, which proves that A is open. The continuity of f is thus proved. ■

Exercise 2°. Let $X = A \cup B$ be the union of two closed sets. Then the mapping $f : X \to Y$ is continuous if and only if the mappings $f|_A$ and $f|_B$ are continuous. Give counterexamples when the condition of the closedness of the sets A and B is not fulfilled.

2. Three Problems Leading to Continuous Mappings.

In topology and its applications, the following types of problems have to be solved often:

(i) Given two topological spaces X, Y and a mapping $f : X \to Y$. Check whether f is continuous.

(ii) Given a topological space X, a set Y and a mapping $f : X \to Y$. Equip Y with a topology so as to make f continuous.

(iii) Given a topological space Y, a set X and a mapping $f : X \to Y$. Equip X with a topology so as to make f continuous.

Problem (i) has already been considered for certain spaces and mappings. To solve it, extra information about X, Y, and f is required.

Problem (ii) can be solved without any additional assumptions. Let $\{U\} = \tau$ be a topology on X. We endow Y with a topology by calling open in Y those and only those sets $V \subset Y$ whose inverse images $f^{-1}(V) = U$ are open in X (including the empty inverse image). It is not difficult to verify that the collection of such sets $\{V\}$ forms a topology. If V_α are some sets from $\{V\}$ then

$$(1) \qquad \bigcup_\alpha V_\alpha \in \{V\}, \text{ since } f^{-1}\left(\bigcup_\alpha V_\alpha\right) = \bigcup_\alpha f^{-1}(V_\alpha)$$

$$= \bigcup_\alpha U_\alpha \in \tau, \text{ where } U_\alpha = f^{-1}(V_\alpha) \in \tau;$$

$$(2) \quad \bigcap_{i=1}^{k} V_{\alpha_i} \in \{V\}, \text{ since } f^{-1}\left(\bigcap_{i=1}^{k} V_{\alpha_i}\right) = \bigcap_{i=1}^{k} f^{-1}(V_{\alpha_i})$$

$$= \bigcap_{i=1}^{k} U_{\alpha_i} \in \tau;$$

(3) $Y \in \{V\}$, since $f^{-1}(Y) = X \in \tau$;
(4) $\varnothing \in \{V\}$, since $f^{-1}(\varnothing) = \varnothing \in \tau$.

The topology constructed will be called the topology *induced by the mapping f*; this is the strongest topology in which f is continuous*.

Consider now the continuity of a mapping $g : X/R \to Y$, where R is a certain equivalence, and X/R is a factor space.

THEOREM 3. *Let $g : X/R \to Y$ and $f : X \to Y$ be two mappings, and $\pi : X \to X/R$ the projection. Let the diagram*

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "f"'] & & X/R \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

be commutative, i.e, $f(x) = (g\pi)(x)$, $x \in X$. *Then g is continuous if and only if f is continuous.*

PROOF. Let f be continuous. Consequently, if $V \subset Y$ is open then $f^{-1}(V)$ is open in X. Owing to the commutativity of the diagram, the set $\pi(f^{-1}(V)) = U$ is open in

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* This method of equipping with a topology was mentioned earlier when we introduced the concept of quotient topology (see Sec. 3).

X/R. Since $f = g\pi$,

$$\pi(f^{-1}(V)) = \pi(\pi^{-1}g^{-1})(V) = g^{-1}(V).$$

Therefore, g is continuous.

Let g be continuous, i.e., $g^{-1}(V)$ is open in X/R if V is open in Y. Then $\pi^{-1}(g^{-1}(V))$ is open in X due to the continuity of π. However, $\pi^{-1}(g^{-1}(V)) = f^{-1}(V)$, and hence f is continuous. ∎

We must find out when the space Y with the topology described above is homeomorphic to the factor space of the space X with respect to the following equivalence relation (induced by f):

$$R_f : x_1 \sim x_2 \Leftrightarrow f(x_1) = f(x_2).$$

The class of equivalent points in X is the full inverse image $f^{-1}(y)$ of some value $y \in Y$. Let $\pi : X \to X/R_f$ be the projection, and $\hat{f} : X/R_f \to Y$ the residue class mapping transforming the class of equivalent points $[x]$ into $f(x)$. We thus have the equality $\hat{f}(\pi(x)) = f(x)$, $x \in X$, which means the commutativity of the diagram

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "f"'] & & X/R_f \arrow[dl, "\hat{f}"] \\ & Y & \end{tikzcd}$$

THEOREM 4. *If a topology on Y is induced by a mapping $f : X \to Y$ and f is surjective, then $\hat{f}$ is a homeomorphism of the spaces X/R_f and Y.*

PROOF. The mapping f is continuous if Y is endowed with the indicated topology. By Theorem 3, $\hat{f}$ is continuous. Since $\hat{f}$ is bijective, it is sufficient to prove the continuity of $\hat{f}^{-1}$, which is equivalent to the openness of $\hat{f}$. To prove it, let U_R be an open set in X/R_f, and $V = \hat{f}(U_R)$ its image in Y. Let $\hat{f} = g$. Then we have

$$f^{-1}(V) = (g\pi)^{-1}(V) = \pi^{-1}(g^{-1}(V)) = \pi^{-1}(U_R).$$

Since the set $\pi^{-1}(U_R)$ is open, $f^{-1}(V)$ is also open; by the definition of the strongest topology on Y in which f is continuous, we make a conclusion that V is open in Y. ∎

Exercise 3°. Show that if a mapping f is not surjective, then the factor space X/R_f is homeomorphic to the subspace $f(X) \subset Y$, where the topology on Y is induced by f.

When considering a continuous mapping $f : X \to Y$ of two topological spaces, a question may be asked under what conditions the topology of Y is induced by the mapping f.

THEOREM 5. *Let $f : X \to Y$ be a surjective mapping of two topological spaces and let f be continuous and open* (*or closed*). *Then the topology on Y is a quotient topology induced by f.*

PROOF. Consider the case when f is open. Let $\{V\} = \tau_1$ be the topology on Y induced by the mapping f, and $\tau_2 = \{U\}$ the original topology on Y. We shall show that they coincide. In fact, let $V \in \tau_1$, $V \neq \varnothing$. Then, since f is surjective, $f^{-1}(V) \neq \varnothing$ and $f^{-1}(V)$ is open in X (by the method of construction of τ_1). Since

f is open, we find that the set $f(f^{-1}(V)) = V$ is open in Y, i.e., $V \in \tau_2$. Conversely, let $U \in \tau_2$; then it follows from the continuity of f that $f^{-1}(U)$ is open in X. Therefore, $U \in \tau_1$ by the definition of the topology τ_1.

The case when f is closed is considered in an analogous way. ∎

It remains for us to consider problem (iii). Let $f : X \to Y$ be a mapping of a set X into a topological space Y. Let $\tau = \{V\}$ be a topology in Y. Put $\sigma = \{f^{-1}(V)\}_{V \in \tau}$. The system σ satisfies the axioms of topology (verify!). It is obvious that f is continuous as a mapping of topological spaces (X, σ), (Y, τ). It is clear also that σ is the weakest of the topologies possessing this property.

9. PRODUCTS OF TOPOLOGICAL SPACES

1. Topology in the Direct Product of Spaces. Obtaining the direct products of topological spaces makes the construction of new topological spaces possible.

Remember that the *direct product* $X \times Y$ of the sets X, Y is the collection of ordered pairs (x, y), where $x \in X$, $y \in Y$. The direct products of any number of factors can be considered. An element of such a product $\prod_{\alpha \in A} X_\alpha$ is a set $\{x_\alpha\}_{\alpha \in A}$, $x_\alpha \in X_\alpha$ or, in other words, the elements $\prod_{\alpha \in A} X_\alpha$ are functions $x : A \to \cup X_\alpha$ such that $x(\alpha) \in X_\alpha$. If $A = \{1, 2, \dots, n\}$ is a finite set then the product of $X_1, X_2, \dots, X_n$ is often denoted by $X_1 \times X_2 \times \dots \times X_n$, its elements being the ordered sets $(x_1, x_2, \dots, x_n)$, where $x_i \in X_i$, $i = 1, 2, \dots, n$.

We endow the direct product $X \times Y$ with a topology by giving the family $\{U_\alpha \times V_\beta\}$ as its base, where $\{U_\alpha\}$, $\{V_\beta\}$ are the bases for the topologies in X and Y, respectively.

Exercise 1°. Verify that the covering $\{U_\alpha \times V_\beta\}$ of the set $X \times Y$ satisfies the criterion of a base (see Sec. 1).

The topology determined by the base $\{U_\alpha \times V_\beta\}$ is called the *product topology*.

EXAMPLE. The plane R^2 is the direct product of straight lines: $R^2 = R^1 \times R^1$. The base for the topology on R^2 is the set of open rectangles of the form $U_\alpha \times V_\beta$, i.e., two-dimensional parallelepipeds (Fig. 52), where U_α, V_β are intervals.

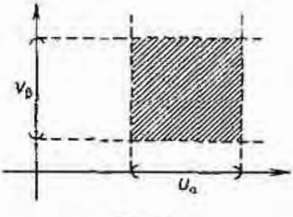

Fig. 52

Exercises.

2°. Prove that the two-dimensional torus T^2 is homeomorphic to the product $S^1 \times S^1$.

3°. Prove that the space $S^1 \times R^1$ is homeomorphic to the circular cylinder.

Consider the projections

$$p_1 : X \times Y \to X, \quad (x, y) \to x; \quad p_2 : X \times Y \to Y, \quad (x, y) \to y.$$

THEOREM 1. *The mappings p_1, p_2 are continuous in the product topology. Moreover, this is the weakest topology in which p_1 and p_2 are continuous.*

PROOF. We show that p_1 is continuous. Let U_α be a set from the base for X. It suffices to show that $p_1^{-1}(U_\alpha)$ is open. Since the space Y can be represented as the union $\bigcup_\beta V_\beta$ of all the sets of the base,

$$p_1^{-1}(U_\alpha) = U_\alpha \times Y = U_\alpha \times \bigcup_\beta V_\beta = \bigcup_\beta (U_\alpha \times V_\beta)$$

and therefore $p_1^{-1}(U_\alpha)$ is open in $X \times Y$. The continuity of p_2 is verified in the same way.

Let us verify the second statement of the theorem. For p_1 to be continuous, it is necessary that the sets $p_1^{-1}(U_\alpha) = U_\alpha \times Y$ should be open. For p_2 to be continuous, the sets $X \times V_\beta = p_2^{-1}(V_\beta)$ should be open. Hence for both p_1 and p_2 to be continuous simultaneously, it is necessary that the sets $U_\alpha \times Y$ and $X \times V_\beta$, and therefore, the sets $(U_\alpha \times Y) \cap (X \times V_\beta) = U_\alpha \times V_\beta$ should be open.

Thus, any topology on $X \times Y$ in which p_1 and p_2 are continuous should contain the sets $U_\alpha \times U_\beta$ (and also the topology generated by them). Therefore, it is stronger than the product topology of $X \times Y$. ■

Consider the direct product $\prod_{\alpha \in A} X_\alpha$ of an arbitrary (possibly, infinite) number of factors. We introduce the weakest of all those topologies on $\prod_{\alpha \in A} X_\alpha$ in which every projection $p_{\alpha'} : \prod_{\alpha \in A} X_\alpha \to X_{\alpha'}$ which associates the function x with the value $x(\alpha')$, is continuous. This topology on $\prod_{\alpha \in A} X_\alpha$ is called the *product topology* or the *Tihonov topology*.

We shall describe it in more detail. The subbase for the Tihonov topology can most easily be characterized as the collection of all the possible sets in the product $\prod_{\alpha \in A} X_\alpha$ which have the form $B_{\alpha_0} = \{x : x(\alpha_0) \subset U_{\alpha_0}\}$, where U_{α_0} is an arbitrary element of the base for the space X_{α_0} and $\alpha_0 \in A$ is also arbitrary. It is easy to see that $B_{\alpha_0} = p_{\alpha_0}^{-1}(U_{\alpha_0})$. Thus, for a certain α_0, the sets $\{B_{\alpha_0}\}$ form the weakest topology on $\prod_{\alpha \in A} X_\alpha$ in which the projection p_{α_0} is continuous. Therefore, having declared the set $\{B_\alpha\}_{\alpha \in A}$ to be the subbase, we obtain the weakest topology in which all the projections $p_{\alpha_0'}$ are continuous.

Hence, the base for the Tihonov topology consists of sets of the form

$$U = p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap p_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \ldots \cap p_{\alpha_n}^{-1}(U_{\alpha_n}).$$

where $\alpha_1, \ldots, \alpha_n$ is an arbitrary finite set of elements $\alpha \in A$, and U_{α_i} is an arbitrary element of the base in X_{α_i}. In other words, an open set of the base is a set of functions

$$\{x : x(\alpha_i) \in U_{\alpha_i}, i = 1, 2, \ldots, n\} = \{x : x(\alpha_1) \in U_{\alpha_1}\} \cap \ldots \cap \{x : x(\alpha_n) \in U_{\alpha_n}\}.$$

THEOREM 2. *For any $\alpha_0 \in A$, the projection $p_{\alpha_0} : \Pi X_\alpha \to X_{\alpha_0}$ is a continuous and open mapping.*

PROOF. The continuity of p_{α_0} has already been verified. Since the p_{α_0}-image of a set from the base for the topology is open, the image of any open set is also open. ■

Exercises.

4°. Show that the base for the Tihonov topology is formed by all possible sets of the form

$$U = \Big(\prod_{\alpha \neq \alpha_1, \ldots, \alpha_n} X_\alpha\Big) \times U_{\alpha_1} \times U_{\alpha_2} \times \ldots \times U_{\alpha_n}$$

(such sets are often referred to as *cylinders*).

5°. Verify that $R^n = \underbrace{R^1 \times \ldots \times R^1}_{n}$. Describe the base and the subbase for the Tihonov topology on R^n.

6°. Verify that the n-dimensional cube I^n in R^n can be represented as $I^n = \underbrace{I \times \ldots \times I}_{n}$, where $I = [0, 1]$.

7°. Consider the n-dimensional torus $T^n = \underbrace{S^1 \times \ldots \times S^1}_{n}$ (there are n factors) and describe the subbase and base for its topology.

2. Continuous Mappings into the Product of Spaces.

We investigate the mappings $f : X \to \prod_{\alpha \in A} X_\alpha$ from a topological space X into a product. We can consider the components $f_\alpha : X \to X_\alpha$ and $f_\alpha = p_\alpha f$ of the mapping f. Conversely, if a set of mappings $\{f_\alpha : X \to X_\alpha, \alpha \in A\}$ is given, then the mapping $f : X \to \prod_{\alpha \in A} X_\alpha$ is determined uniquely.

Thus, there exists a bijection between the set of mappings $f : X \to \prod_{\alpha \in A} X_\alpha$ and the family of the sets of mappings $\{f_\alpha\}_{\alpha \in A}$.

THEOREM 3. (*A mapping f is continuous*) $\Leftrightarrow$ (*the mapping f_α is continuous for each $\alpha \in A$*).

PROOF. Let all f_α be continuous. In order to show that f is continuous, it suffices to show that $f^{-1}(U)$ is open in X for any U from the base for $\prod_{\alpha \in A} X_\alpha$. Let

$$U = \Big(\prod_{\alpha \neq \alpha_1, \ldots, \alpha_k} X_\alpha\Big) \times U_{\alpha_1} \times \ldots \times U_{\alpha_k},$$

then

$$f^{-1}(U) = \{x \in X : f_\alpha(x) \in X_\alpha, \alpha \neq \alpha_1, \ldots, \alpha_k, f_{\alpha_i}(x) \in U_{\alpha_i}, \ i = 1, 2, \ldots, k\}$$
$$= \bigcap_{\alpha \neq \alpha_1, \ldots, \alpha_k} f_\alpha^{-1}(X_\alpha) \cap f_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \ldots \cap f_{\alpha_k}^{-1}(U_{\alpha_k})$$
$$= X \cap V_1 \cap \ldots \cap V_k,$$

where $V_i = f_{\alpha_i}^{-1}(U_{\alpha_i})$ is an open set in X owing to the continuity of f_{α_i}. Therefore, $f^{-1}(U)$ is open in X. The proof of the converse is left to the reader. ■

Consider now the mapping $f\colon \prod_{\alpha \in A} X_\alpha \to X$ associating each family $\{x(\alpha)\}_{\alpha \in A}$ with the corresponding element from X.

Exercise 8°. Verify that if $X = X_\alpha = R^1$, $\alpha \in A$, $A = \{1, 2, \ldots, n\}$, then the mapping f is a numerical function in n arguments.

In the general case, the mapping f can be considered to be a generalization of a numerical function in n arguments, assuming that it depends on the variables $x(\alpha) \in X_\alpha$. If all values $x(\alpha)$ are fixed (except $x(\alpha_0)$) then a function in one argument is obtained, the argument varying in X_{α_0}. We make these ideas more precise.

Consider a subspace X'_{α_0} of the product consisting of all functions x taking the value $x(\alpha) = y_\alpha$, $\alpha \neq \alpha_0$, where $y_\alpha \in X_\alpha$ is a certain element.

Exercise 9°. Verify that X'_{α_0} is homeomorphic to X_{α_0}.

Let $\pi_{\alpha_0}\colon X_{\alpha_0} \to X'_{\alpha_0}$ be the natural homeomorphism (depending on certain y_α, $\alpha \neq \alpha_0$), and $f|_{X'_{\alpha_0}}$ the restriction of f to X'_{α_0}. The diagram

$$\begin{tikzcd} X'_{\alpha_0} \arrow[r, "f|X'_{\alpha_0}"] & X \\ X_{\alpha_0} \arrow[u, "\pi_{\alpha_0}"] \arrow[ur, dashed] & \end{tikzcd}$$

can be naturally completed to a commutative one by the product of two mappings (see the dotted arrow). We denote it by $f_{\alpha_0}^{\{y_\alpha\}}$. This product of the two mappings characterizes the dependence of f on the argument $x(\alpha_0) \in X_{\alpha_0}$ for the given values y_α of the other arguments $x(\alpha)$.

Exercise 10°. Verify that if f is continuous, then the mapping $f_{\alpha_0}^{\{y_\alpha\}}\colon X_{\alpha_0} \to X$ is continuous for all $\alpha_0 \in A$, $y_\alpha \in X_\alpha$, when $\alpha \neq \alpha_0$. The converse is not true. Give an example.

We shall consider another case of a mapping of the products of topological spaces. Let $f_\alpha\colon X_\alpha \to Y_\alpha$, $\alpha \in A$ be some collection of mappings of topological spaces. The mapping $\prod_{\alpha \in A} f_\alpha\colon \prod_{\alpha \in A} X_\alpha \to \prod_{\alpha \in A} Y_\alpha$ is determined naturally if each function $x \in \prod_{\alpha \in A} X_\alpha$ is associated with a function $y \in \prod_{\alpha \in A} Y_\alpha$ by the rule $y(\alpha) = f_\alpha(x(\alpha))$. This mapping is called the *product of mappings* f_α. When $A = \{1, 2, \ldots, n\}$, the product of the mappings $f_1, f_2, \ldots, f_n$ is often denoted by

$$f_1 \times f_2 \times \ldots \times f_n\colon X_1 \times X_2 \times \ldots \times X_n \to Y_1 \times Y_2 \times \ldots \times Y_n.$$

Exercises.

11°. Prove that ($\prod_{\alpha \in A} f_\alpha$ is continuous) $\Leftrightarrow$ (f_α is continuous for each $\alpha \in A$).

12°. The *graph of a mapping* $f\colon X \to Y$ is the subset $\Gamma_f \subset X \times Y$ of the form $\Gamma_f = \{(x, y) : x \in X, y = f(x)\}$. Verify that

(1) Γ_f is the image of the mapping $\bar{f}\colon X \to X \times Y$, $\bar{f}(x) = (x, f(x))$;
(2) (f is continuous) $\Leftrightarrow$ ($\bar{f}$ is continuous);
(3) (f is continuous) $\Leftrightarrow$ (Γ_f is closed).

13°. R-equivalent points in $X(x \overset{R}{\sim} y)$ can be combined into a pair $(x, y) \in X \times X$; denote the set of all such pairs by $R \subset X \times X$. Show that (1) if X/R is Hausdorff, then the set R is closed; (2) if the projection $\pi : X \to X/R$ is open and the set R is closed, then X/R is Hausdorff.
14°. Show that the product of Hausdorff spaces is also a Hausdorff space.
15°. Show that a space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x)\}$ is closed in $X \times X$.

10. CONNECTEDNESS OF TOPOLOGICAL SPACES

1. The Concept of Connectedness of a Topological Space. The idea of connectedness generalizes an intuitive idea of the wholeness or unseparatedness of a geometric figure, and the idea of a disconnected space generalizes that of the negation of wholeness, i.e., separatedness. These ideas admit a strict definition within the theory of topological spaces and are studied in detail in the present section.

Consider a topological space X and its subsets A, B.

DEFINITION 1. The sets A and B are said to be *separated* if

$$\bar{A} \cap B = A \cap \bar{B} = \varnothing.$$

For example, if X is R^1, i.e., the number line and $A = (a, b)$, $B = (b, c)$ are intervals, $a < b < c$, then A and B are separated; but if $A = (a, b]$ and $B = (b, c)$, then A and B are not separated (i.e., $A \cap \bar{B} = \{b\}$).

DEFINITION 2. A space X is said to be *disconnected* if it can be represented as the union of two nonempty separated sets.

A space not satisfying the condition of Definition 2 is said to be *connected*. Thus, a connected space cannot be represented as the union of two nonempty separated sets.

We can speak of the connectedness (resp. disconnectedness) of a subset A of a topological space X by considering A to be a topological space with the induced topology.

The simplest examples of connected spaces are:

(1) a one-point space $X = \{*\}$; (2) an arbitrary set X with the topology τ_0 (i.e., trivial). The simplest example of a disconnected space is a two-point space X with the discrete topology (verify that the definitions are valid!).

We shall adduce another definition of a disconnected space which is used quite often.

DEFINITION 3. A topological space X is said to be *disconnected* if it decomposes into the union of two nonempty, disjoint, open sets.

Note that two mutually complementary, open (resp. closed) sets are simultaneously closed (resp. open).

We shall prove the equivalence of Definitions 2 and 3. (1) Let X be disconnected in the sense of Definition 2. Then we have $X = A \cup B$, where $\bar{A} \cap B = \varnothing$,

$A \cap \bar{B} = \varnothing$ and A and B are nonempty. Therefore, $\bar{A} \subset X \setminus B$ and $\bar{B} \subset X \setminus A$, i.e., $\bar{A} = A$, $\bar{B} = B$, which implies the closedness of A and B. The conditions of Definition 3 are thus fulfilled.

(2) Conversely, let X be disconnected in the sense of Definition 3. Then $X = A \cup B$, A, B are nonempty and open, $A \cap B = \varnothing$. It is obvious that A and B are closed. Hence, $\bar{A} \cap B = \varnothing$, since $A = \bar{A}$; $\bar{B} \cap A = \varnothing$ because $\bar{B} = B$. ■

The following theorem produces an important example of a connected space.

THEOREM 1. *Any segment $[a, b]$ of the number line R^1 is connected.*

PROOF. Consider a topological space $X = [a, b]$ equipped with the topology from R^1. Assume that X is disconnected: $X = U \cup V$, $U \cap V = \varnothing$, where U, V are nonempty and open (and simultaneously closed). Let $a \in U$. We will consider line-segments $[a, x]$ where $x \in (a, b]$.

When x is near to a, $[a, x] \subset U$ because U is open. Denote the supremum of x such that $[a, x] \subset U$ by a_* ($a_* \in X$); it is clear that $a_* \neq b$.

If $a_* \in U$ then points near to a_* (on the left and on the right) also lie in U, which is contrary to the definition of a_*. Therefore, $a_* \notin U$. However, by the definition of a_*, the inclusion relation $a_* \in V$ is impossible. Thus, we have run up against a contradiction with the equality $X = U \cup V$. ■

Now the connectedness of more general spaces can be established.

THEOREM 2. *Any convex set $T \subset R^n$ is connected; in particular, the space R^n itself is connected.*

PROOF. Let $T = U \cup V$, where U and V are nonempty separated open sets. Let $[a, b] = X$ be the line-segment joining some points $a \in U$ and $b \in V$. Then $X \cap U = U_X \neq \varnothing$, $X \cap V = V_X \neq \varnothing$ and $X = U_X \cup V_X$ is the decomposition of X, which is contrary to the connectedness of the segment X. ■

As an example of a disconnected set, consider the set of rational numbers $Q = \left\{\frac{p}{q}\right\}$ on the straight line R^1. Let $\alpha \in R^1$ be an arbitrary irrational number. Then the sets

$$U_X = \{x : x \in Q, x < \alpha\} \text{ and } V_X = \{x : x \in Q, x > \alpha\}$$

are nonempty, open and disjoint. Thus, the decomposition $X = U_X \cup V_X$ means that $X = Q$ is disconnected. ■

Exercises.

1°. Prove that the set of all irrational numbers is disconnected.

2°. (a) Show that the set $\bar{A}$ is connected if A is connected; (b) show that in a space endowed with the discrete topology, any set (except one-point sets) is disconnected.

2. Properties of Connected Spaces. Note first that connectedness (resp. disconnectedness) is a topological property of the space, i.e., it is preserved under homeomorphisms. In fact, this follows from the fact that the separatedness of sets is preserved under homeomorphisms.

Connectedness is preserved in a more general way under continuous mappings.

THEOREM 3. *Let $f : X \to Y$ be a continuous mapping. If X is connected then $f(X)$ is connected in Y.*

PROOF. Assume the contrary, i.e., $f(X) = U_1 \cup V_1$, where $U_1 \cap V_1 = \varnothing$, U_1, V_1 being open in $f(X)$, $U_1 \neq \varnothing$ and $V_1 \neq \varnothing$. That $f(X)$ is open implies that there exist two sets U and V that are open in Y and such that $U \cap f(X) = U_1$, $V \cap f(X) = V_1$. But, clearly, $f(X) = U_1 \cup V_1 \Rightarrow X = f^{-1}(U_1) \cup f^{-1}(V_1)$ is the union of nonempty, disjoint subsets. Since $f^{-1}(U_1) = f^{-1}(U), f^{-1}(V_1) = f^{-1}(V)$ (why?) and $f^{-1}(U), f^{-1}(V)$ are open sets (due to the continuity of f), the decomposition $X = f^{-1}(U_1) \cup f^{-1}(V_1)$ is contrary to the connectedness of X. ■

Exercise 3°. (a) Show that the graph Γ_f of a continuous mapping f of a connected space is connected.

(b) Hence deduce the theorem that the numerical continuous function $f : [0, 1] \to R$ has a zero $\xi : f(\xi) = 0$ in the interval $(0, 1)$ if $f(0)\cdot f(1) < 0$.

Statement (b) of Exercise 3 is a variant of the classical Bolzano theorem proved in analysis. To the Bolzano theorem, a more general intermediate-value theorem is related, viz., if a numerical function $f(x)$ is continuous on a line-segment $[a, b]$, $f(a) \neq f(b)$, and a number C is included between the numbers $f(a)$ and $f(b)$, then there exists a point $c \in [a, b]$ such that $f(c) = C$.

This theorem also follows from Theorem 3. In fact, the intermediate-value theorem is equivalent to the nonempty intersection of the graph Γ_f of a numerical function $f(x)$ with the straight line $y = C$ in the plane R^2, which follows from the connectedness of the graph Γ_f and the choice of the number C.

The Bolzano and the intermediate-value theorem could have been proved without resorting to the graph of the mapping f. Instead, the proof could have relied on the connectedness (in the space R^1) of the image $f([a, b])$ and the property of connected sets in R^1 to contain every intermediate point together with and between any two points (prove!).

Exercise 4°. Prove that the circumference S^1 is connected

Hint: Consider the mapping $[0, 1] \to S^1$ given by the formulae $x = \cos 2\pi t, y = \sin 2\pi t$.

The following theorem is intuitively obvious.

THEOREM 4. *A space X is connected if any two of its points can be 'joined' by some connected subset (i.e., they lie in a connected subset).*

PROOF Assume the contrary. Then $X = U \cup V$ is the corresponding decomposition into open parts ($U \cap V = \varnothing$). Let $u_0 \in U$ and $v_0 \in V$ be two points, and $L \subset X$ a connected set containing u_0 and v_0. Put $U_1 = U \cap L$ and $V_1 = V \cap L$. They are open (and nonempty) sets in L, moreover, $L = U_1 \cup V_1$ and $U_1 \cap V_1 = \varnothing$, but this is contrary to the connectedness of L. ■

Exercise 5°. Verify that: (a) $A \cup B$ is connected if $A, B \subset X$ are connected sets in X, and $A \cap B \neq \varnothing$; and (b) $A \cup B \cup C$ is connected if $A, B, C \subset X$ are connected and $A \cap C \neq \varnothing, B \cap C \neq \varnothing$.

It follows from Exercise 5 that, for example, the sphere $S^n, n \geqslant 1$ is connected. In fact, S^n consists of two closed hemispheres $\bar{S}^n_+$ and $\bar{S}^n_-$ whose intersection is the

equatorial sphere S^{n-1}, and each hemisphere is connected as it is a continuous image of the disc (see Sec. 2). ■

We shall now establish the following more general criterion of connectedness.

THEOREM 5. *Given a family of sets* $\{A_\alpha\}$ *that are connected in X and pairwise unseparated, then* $C = \bigcup_\alpha A_\alpha$ *is connected in X.*

PROOF. Assume the contrary: let $C = D_1 \cup D_2$, $D_1 \cap D_2 = \varnothing$, and D_1, D_2 be nonempty and closed in C. For an arbitrary A_α, the following cases arise:

(1) $A_\alpha \subset D_1$, (2) $A_\alpha \subset D_2$, (3) $A_\alpha \cap D_1 \neq \varnothing$, $A_\alpha \cap D_2 \neq \varnothing$.

However, case (3) can be excluded due to the connectedness of A_α. Hence, we have the sets $A_{\alpha_i} \subset D_i$, $i = 1, 2$, but the closedness of D_i in C implies that $(\bar{A}_{\alpha_i} \cap C) \subset D_i$, $i = 1, 2$.

It is evident that $(\bar{A}_{\alpha_1} \cap C) \cap A_{\alpha_2} = \varnothing$, $A_{\alpha_1} \cap (\bar{A}_{\alpha_2} \cap C) = \varnothing$, and by taking into account the inclusion relations $A_{\alpha_i} \subset C$, $i = 1, 2$, we obtain that $\bar{A}_{\alpha_1} \cap A_{\alpha_2} = \varnothing$ and $A_{\alpha_1} \cap \bar{A}_{\alpha_2} = \varnothing$, which is contrary to the assumption that A_{α_1} and A_{α_2} are not separated. ■

One special class of spaces satisfies Theorem 4. They are termed path-connected spaces. To describe them, we introduce the concept of path in X.

DEFINITION 4. The continuous mapping s: $[0, 1] \to X$, $s(0) = a$, $s(1) = b$ is called a *path* connecting two points a and b of a topological space X.

Exercise 6°. Verify that the image $s(I)$ of the line-segment $I = [0, 1]$ is a connected set connecting the points a and b.

DEFINITION 5. A space X is said to be *path-connected* if any two points in it can be connected by a path.

An example of a path-connected space may be given by a closed surface (see Sec. 4).

It follows from Theorem 4 that a path-connected space is necessarily connected. That the converse is not valid can be demonstrated by the following example. Consider the union of sets in R^2:

$$X = [(0, 0), (1, 0)] \bigcup_{n=1}^{\infty} \left[\left(\frac{1}{n}, 0\right), \left(\frac{1}{n}, 1\right)\right] \cup (0, 1),$$

and denote the line-segment connecting the points P and Q in R^2 by $[P, Q]$; X is connected but not path-connected (the point $(0, 1)$ cannot be connected by a path with any other point from X).

Exercises.

7°. Verify that convex sets in R^n and the sphere S^n, $n \geqslant 1$, are path-connected.

8°. Prove that if $A \subset X$ is connected, then any B such that $A \subset B \subset \bar{A}$ is also connected. Give examples.

Finally, we shall consider the product of connected spaces.

THEOREM 6. *The product* $X \times Y$ *of connected spaces is connected.*

PROOF. Assume the contrary. Let $X \times Y = U \cup V$ be a decomposition into open

(nonempty) sets, $U \cap V = \varnothing$. Let $(x_0, y_0) \in U$. The set $x_0 \times Y$ is homeomorphic to Y and therefore connected; intersecting U at the point (x_0, y_0), it lies wholly in U, which follows from the connectedness of U. The sets $X \times y$, $y \in Y$, intersect $x_0 \times Y$ and therefore U. However, being connected, they lie wholly in U. Thus, $\bigcup_{y \in Y} (X \times y) = X \times Y \subset U$. Therefore, $V = \varnothing$. The contradiction proves the theorem. ■

Exercises.

9°. Prove Theorem 6 for the product of n connected spaces ($n > 2$).

10°. Prove the connectedness of the Tihonov product $\prod_{\alpha \in A} X_\alpha = Y$ of connected spaces X_α.

Hint: Consider the set R of the points of the product that can be joined to a certain point by connected sets, and verify that $R = Y$.

3. **Connected Components.** If a space is disconnected then it is natural to attempt to decompose it into connected pieces. We describe this decomposition. Let $x \in X$ be a point in a topological space X. Consider the largest connected set containing the point x: $L_x = \bigcup A_x$, where all A_x are connected sets containing the point x. The set L_x is closed since the closure $\bar{L}_x$ of the connected set L_x is connected (see Exercise 2) and hence $\bar{L}_x \subset L_x$, i.e., $\bar{L}_x = L_x$.

DEFINITION 6. The set L_x is called the *connected component* of a point x in a topological space X.

Let $x, y \in X$, $x \neq y$. Consider the sets L_x, L_y. Owing to their connectedness and maximality, there are two possibilities: either (1) $L_x = L_y$ or (2) $L_x \cap L_y = \varnothing$. In the second case, L_x is separated from L_y, since $L_x \cap \bar{L}_y = \varnothing$, $L_y \cap \bar{L}_x = \varnothing$. In the evident equality $X = \bigcup L_x$, where the union is taken over all $x \in X$, we reject all repeated components.

Thus, the following theorem has been proved.

THEOREM 7. *Any topological space can be decomposed into the union of its connected components which are closed and disjoint.*

NOTE. Generally speaking, connected components cannot also be open (give examples).

Exercises.

11°. Verify that if a space X possesses a finite number of connected components, then they are open.

12°. Verify that the number of connected components of a space X (either finite or understood as the power of the set) is a topological characteristic of the space.

11. COUNTABILITY AND SEPARATION AXIOMS

The topological spaces usually encountered in various mathematical problems possess other properties. A number of these properties are expressed by what are called the countability and separation axioms.

1. Countability Axioms. In Sec. 1, the notion of base for a topology was introduced. It becomes clear, while investigating topological spaces, that spaces possessing a countable base for a topology have a number of useful properties. Hence, the following definition is introduced.

DEFINITION 1. A topological space (X, τ) is said to satisfy the *second countability axiom* if its topology possesses a countable base.

EXAMPLE 1. The spaces R^1 and R^n satisfy the second countability axiom. ♦

It is interesting to compare spaces satisfying the second countability axiom, and separable spaces. It turns out that a separable space does not necessarilly satisfy the second countability axiom.

EXAMPLE 2. Consider an uncountable set X whose topology consists of the complements to all possible finite subsets of the set X, the whole set X and the empty set. (Verify that such a set of subsets really forms a topology.) Any infinite subset in this space is dense since it intersects each open set. This implies the separability of X. On the other hand, assume that there is a countable base $\mathscr{B}$ in X. Consequently, if $x \in X$ is a fixed point then the intersection of all the open sets containing x equals $\{x\}$. Therefore, the countable intersection of all those elements of the base that contain x also equals $\{x\}$. But then the complement $X \setminus \{x\}$ is the union of at most a countable set of finite sets and hence is at most countable too. This is contrary to the assumption that X is uncountable. ♦

On the other hand, the following theorem is valid.

THEOREM 1. *A space satisfying the second countability axiom is separable.*

We leave the proof of this theorem to the reader.

It is important to note that the converse statement that is, generally speaking, incorrect (which has just been demonstrated by way of an example) happens to hold for metric spaces. Viz., the following theorem is valid.

THEOREM 2. *Any separable metric space (M, ρ) satisfies the second countability axiom.*

PROOF. Let $A = \{a_1, a_2, \ldots, a_n, \ldots\}$ be a countable and everywhere dense set in M. Take the collection of open sets

$$\mathscr{B} = \left\{ V_{n,k} = \left\{ x : \rho(x, a_n) < \frac{1}{k} \right\} ; n, k = 1, 2, \ldots \right\}$$

as a base. It is easy to see that this is indeed a base.

In fact, the separability of M means that for any $x \in M$, in any ball $D_\varepsilon(x) = \{y : \rho(x, y) < \varepsilon\}$, there is an element $a_n \in A$; moreover, there is a number k such that $V_{n,k} \subset D_\varepsilon(x)$. Since any open set in M can be represented as the union of balls, it can also be represented as the union of sets $V_{n,k}$ from $\mathscr{B}$ ■

We need the following statement for the future as well.

THEOREM 3. (LINDELÖF'S THEOREM). *If a space X satisfies the second countability axiom then in its arbitrary open covering $\{U_\alpha\}$, there is at most a countable subcovering.*

PROOF. Let $\mathscr{B}$ be a countable base for the topology on X. Since any element of th covering $\{U_\alpha\}$ is the union of sets from $\mathscr{B}$, a subfamily C can be singled out in $\mathscr{B}$, which is at most countable, also covers X and such that each element of C is contained in some element of the family $\{U_\alpha\}$. Then, having chosen for each element of the covering C a set from $\{U_\alpha\}$ containing it, we obtain at most a countable subcovering of the covering $\{U_\alpha\}$. ■

Besides the base for the topology introduced in Sec. 1, there is the important concept of base for the neighbourhood system of a point x of a topological space X.

DEFINITION 2. The family $\{B(x)\}$ of the neighbourhoods of a point x is called the *base for the neighbourhood system* of x if there is a neighbourhood from this family in each neighbourhood of the point x.

Thus, the family of all open neighbourhoods of a point is a base for the neighbourhood system of the point.

EXAMPLE 3. Let $X = M$ be a metric space, and

$$B(x) = \left\{ V^k(x) = \left\{ y : \rho(y, x) < \frac{1}{k} \right\} \right\}_{k=1}^{\infty}$$

a base for the neighbourhood system of the point x. A spherical neighbourhood can be made a refinement of any neighbourhood of x. For any spherical neighbourhood $D_\varepsilon(x) = \{y : \rho(x, y) < \varepsilon\}$, a number k can be chosen such that $\frac{1}{k} < \varepsilon$; then $V^k(x) \subset D_\varepsilon(x)$.

DEFINITION 3. A space X is said to satisfy the *first countability axiom* if the neighbourhood system of any of its points possesses a countable base.

EXAMPLES.

4. A metric space is a first countable space.
5. The space of continuous functions $C_{[0, 1]}$ satisfies the first countability axiom. ♦

Does the space $C_{[0, 1]}$ satisfy the second countability axiom? That it does follows from $C_{[0, 1]}$ being separable, and also from Theorem 2 of this section.

The separability of the space $C_{[0, 1]}$ follows from the Weierstrass theorem which states that any continuous function on the line-segment $[0, 1]$ can be uniformly approximated by a polynomial to any degree of accuracy. Thus, a countable and everywhere dense set A in $C_{[0, 1]}$ consists of the set of all polynomials $\{P_n(t)\}$ with rational coefficients.

Exercise 1°. Verify that a space satisfying the second countability axiom also satisfies the first.

That the converse is not true is demonstrated by the following example.

EXAMPLE 6. Any uncountable space X with the discrete topology satisfies the first countability axiom. In fact, any point $x \in X$ possesses a base for the neighbourhood system consisting of a single neighbourhood $V = \{x\}$. But such a space does not satisfy the second countability axiom. This follows from Lindelöf's theorem and from the fact that the covering formed by one-point sets $\{x\}$, $x \in X$, has no countable subcovering.

Thus, the fulfilment of the second countability axiom is a stronger condition on a topological space than is the first countability axiom.

2. **Properties of Space Separation.** Some important topological properties are characterized by the separation axioms. These axioms enable us to restrict the class of the spaces studied in order to consider their deeper properties.

We shall adduce the main *separation axioms* T_0-T_4. A topological space X is said to be a *T_i-space* if an axiom T_i, $i = 0, 1, 2, 3, 4$, is fulfilled for it.

(T_0). *At least one of any two different points in a space possesses a neighbourhood which does not contain the other point.*

(T_1). *Each point of any pair of different points possesses a neighbourhood which does not contain the other point.*

Any one-point set in a T_1-space is closed.

(T_2). *For any two points $x, y \in X$, $x \neq y$, there exist their neighbourhoods $U(x)$, $U(y)$ such that $U(x) \cap U(y) = \varnothing$.* In this case, X is said to be a *Hausdorff space* (see also Sec. 1).

In order to proceed further, we must introduce the concept of neighbourhood of a set in a topological space.

DEFINITION 4. *A neighbourhood of a set A* in a topological space X is any open set U containing A.

(T_3). *For any point $x \in X$ and any closed set F from X, $x \bar{\in} F$, there exist neighbourhoods $U(x)$, $U(F)$ such that $U(X) \cap U(F) = \varnothing$.*

If the axioms T_1 and T_3 are fulfilled simultaneously then the space X is called a *regular space*.

(T_4). *For any two closed subsets F_1, $F_2 \subset X$, $F_1 \cap F_2 = \varnothing$, there exist neighbourhoods $U(F_1)$, $U(F_2)$ such that $U(F_1) \cap U(F_2) = \varnothing$.*

A space X is called a *normal space* if the axioms T_1 and T_4 are both fulfilled.

We must emphasize that each subsequent axiom from T_0 to T_2 is a stronger condition on the space than the previous. The same is true for the axioms T_2-T_4 but only if the axiom T_1 is fulfilled since T_1 does not follow from either T_3 or T_4. We give some examples.

EXAMPLES.

7. Let R^1 be the real straight line equipped with the topology whose base is formed by rays of the form $a < x < +\infty$. It is not complicated to show that in such a topology, the space R^1 satisfies the axiom T_0 but does not even satisfy the axiom T_1. There is an example of a T_0-space which is not a T_1-space in Exercise 2, Sec. 6, too.

8. Consider the line segment $[0, 1]$ equipped with the topology whose open sets are the empty set and all the sets obtained from $[0, 1]$ by discarding either a finite or countable number of points. This space satisfies the axiom T_1, but is not Hausdorff, i.e., the axiom T_2 is not fulfilled.

We introduce the following definition:

9. In the line-segment $[0, 1]$, the neighbourhoods of an arbitrary point, except zero, are ordinary neighbourhoods and all the possible half-intervals $[0, \alpha)$ with the discarded points $\dfrac{1}{n}$, $n = 1, 2, \ldots$, are called the *neighbourhoods of the zero*

point. It is easy to see that this space is Hausdorff but not regular, because the disjoint closed set $E = \left\{\frac{1}{n} : n = 1, 2, \ldots\right\}$ and the zero point are not separated in the sense of the axiom T_3.

These examples demonstrate that the class of regular (and also normal) spaces is essentially narrower than that of the Hausdorff spaces. However, the class of normal spaces is quite wide and includes, for example, all metric spaces as will be shown in the next section.

Exercises.

2°. Show that in a T_1-space, for any subset A, the following inclusion relation is fulfilled: $(A')' \subset A'$.

3°. Verify that a closed surface (see Sec. 4, Ch. II) is a Hausdorff space.

Note in addition that the difference between normal and regular spaces is quite slight, which is demonstrated by the following.

THEOREM 4 (VEDENISOV). *Any regular space satisfying the second countability axiom is normal.*

The proof of this theorem is not given here.

12. NORMAL SPACES AND FUNCTIONAL SEPARABILITY

1. An Equivalent Definition of Normal Spaces.

The property of normal spaces that is enunciated in the following lemma and which can be assumed to be an equivalent definition of a normal space is useful quite often.

MINOR URYSON LEMMA. *A space X is normal if and only if for any closed set $F \subset X$ and any of its neighbourhoods U, there exists a neighbourhood U' of the set F such that $\overline{U'} \subset U$.*

PROOF. Let X be normal. Consider two closed sets F and $F_1 = X \setminus U$. Since the space is normal, there exist two disjoint neighbourhoods U' and U_1' of the two sets F and F_1. It is clear then that $\overline{U'} \cap F_1 = \varnothing$, whence $\overline{U'} \subset U$. Conversely, let the lemma be fulfilled, and let F_1, F_2 be disjoint closed sets. Consider the set $U_1 = X \setminus F_2$. Then $F_1 \subset U_1$, and it follows from the data given that there exists a neighbourhood U_1' of F_1 such that $\overline{U_1'} \subset U_1$. Having set $U_2 = X \setminus \overline{U_1'}$, we obtain an open set U_2, $F_2 \subset U_2$, whereas $U_1' \cap U_2 = \varnothing$. ∎

COROLLARY. *In a normal space X, two disjoint closed sets F_1, F_2 possess neighbourhoods U_1, U_2 such that $\overline{U_1} \cap \overline{U_2} = \varnothing$.*

Generally speaking, the normality of a subspace does not follow from the normality of the space. However, if any subspace in a normal space X is normal then X is said to be hereditarily normal. The conditions for hereditary normality are given by the following theorem.

THEOREM 1 (URYSON). *A space is hereditarily normal if and only if any two of its separated sets possess disjoint neighbourhoods.*

We do not prove this theorem here.

The image of a normal space under a continuous mapping is not necessarily normal. The simplest example is the identity mapping of the straight line R^1 endowed with the usual topology into the same straight line with some non-Hausdorff topology, for example, trivial. However, there exist sufficient tests for the image of a normal space to be normal. For example, the following statement is valid.

THEOREM 2. *Let X be a normal space and $f : X \to Y$ a continuous closed surjective mapping. Then the space Y is also normal.*

PROOF. Let X be a normal space, $A \subset Y$ a closed subset, and $f^{-1}(A) = A'$. Then the set A' is closed due to the continuity of f. Let U be a neighbourhood of A in Y. Then the set $f^{-1}(U) = U'$ is open (due to the continuity of f) and contains A'. Therefore, U' is a neighbourhood of A', and by the minor Uryson lemma, there exists a neighbourhood V' of the set A' such that $\bar{V}' \subset U'$.

We thus obtain the inclusion relations: $A' \subset V' \subset \bar{V}' \subset U'$. A closed surjective mapping is open. Therefore, $f(V')$ is open, $f(\bar{V}')$ is closed, and we have the relations:

$$A = f(A') \subset f(V') \subset f(\bar{V}') \subset f(U') = U,$$

hence, the normality of Y follows easily. ■

2. **Functional Separability. The Uryson Theorems on Extending Numerical Functions.** The separation of sets was defined above in terms of 'neighbourhoods'. Uryson introduced another notion of separation, or what is called functional separability. It is quite a convenient concept for studying normal spaces.

DEFINITION. Two sets A, B in a topological space X are said to be *functionally separable* if there exists a continuous numerical function $\varphi : X \to R^1$ such that

$$\varphi(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$$

and $0 \leqslant \varphi(x) \leqslant 1$ at all points of X (Fig. 53).

The close relationships between the two concepts of separation are shown up by the following simple lemma.

LEMMA. *If two sets A and B are functionally separable in a topological space then they have disjoint neighbourhoods.*

The proof is left to the reader.

Thus, the functional separability of any pair of closed disjoint sets of a T_1-space implies its normality. It is interesting that the converse statement is also true!

MAJOR URYSON LEMMA. *For any two closed, disjoint sets of a normal space X, there exists a continuous function $\varphi : X \to R^1$ such that $\varphi|_A = 0$, $\varphi|_B = 1$ and $0 \leqslant \varphi(x) \leqslant 1$ for any $x \in X$.*

PROOF. Let A and B be two arbitrary closed sets in X, $A \cap B = \varnothing$. We associate each rational number of the form $r = k/2^n$, where $k = 0, 1, \ldots, 2^n$, with an open set $G(r)$ so that the following properties are fulfilled:

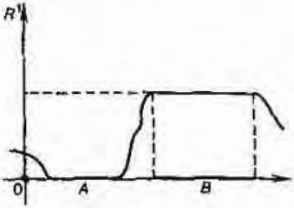

Fig. 53

(1) $A \subset G(0)$, $X \setminus B = G(1)$;
(2) $\overline{G(r)} \subset G(r')$ if $r < r'$.

The proof of the existence of such a family of open sets will be performed by induction on the index n. Let $n = 0$. Since X is normal, there exist disjoint neighbourhoods $U(A)$, $U(B)$ of the sets A and B. Put $G(0) = U(A)$, $G(1) = X \setminus B$ and assume that such a family of sets $G(r)$ is constructed for an index $n - 1$. We will now construct it for the index n. Since $2m/2^n = m/2^{n-1}$, it suffices to construct $G(r)$ for $r = k/2^n$ for an odd k.

Let $k = 2m + 1$, then $(k + 1)/2^n = (m + 1)/2^{n-1}$, $(k - 1)/2^n = m/2^{n-1}$, and therefore by the inductive hypothesis, we already have the inclusion relation $\overline{G((k-1)/2^n)} \subset G((k + 1)/2^n)$. It is evident that the sets $\overline{G((k-1)/2^n)}$ and $X \setminus G((k + 1)/2^n)$ are closed and disjoint. The normality of X means that there exists a neighbourhood V of the set $\overline{G((k-1)/2^n)}$, which does not intersect a certain neighbourhood of the set $X \setminus G((k + 1)/2^n)$. Put $V = G(k/2^n)$. It is clear that

$$\overline{G((k-1)/2^n)} \subset \overline{G(k/2^n)} \text{ and } \overline{G(k/2^n)} \subset G((k+1)/2^n).$$

The induction argument is complete.

We extend the domain of the sets $G(r)$, having put

$$G(r) = \begin{cases} \varnothing & \text{if } r < 0, \\ X & \text{if } r > 1. \end{cases}$$

Now, we specify the function φ as follows: $\varphi(x) = 0$, $x \in G(0)$ and $\varphi(x) = \sup\{r : x \in X \setminus G(r)\}$. We have to show the continuity of φ. To this end, we construct a neighbourhood $U_N(x_0)$ for each point $x_0 \in X$ and each $N > 0$, such that $|\varphi(x_0) - \varphi(x)| < 1/2^N$, $x \in U_N(x_0)$. Let r_0 (of the form $k/2^n$) be such that

$$\varphi(x_0) < r_0 < \varphi(x_0) + 1/2^{N+1}. \tag{1}$$

Put $U_N(x_0) = G(r_0) \setminus \overline{G(r_0 - 1/2^N)}$. Then $x_0 \in U_N(x_0)$ since $r_0 > \varphi(x_0)$ and $r_0 - 1/2^{N+1} < \varphi(x_0)$. If $x \in U_N(x_0)$ then $x \in G(r_0)$. Therefore $\varphi(x) \leqslant r_0$. Furthermore,

$$x \in X \setminus \overline{G(r_0 - 1/2^N)} \subset X \setminus G(r_0 - 1/2^N),$$

therefore $r_0 - 1/2^N \leqslant \varphi(x)$. Thus,

$$r_0 - 1/2^N \leqslant \varphi(x) \leqslant r_0. \tag{2}$$

By comparing (1) and (2), we obtain

$$|\varphi(x_0) - \varphi(x)| < 1/2^N, \quad x \in U_N(x_0),$$

which means that φ is continuous.

It is clear from the method of construction that $\varphi|_A \equiv 0$, $\varphi|_B \equiv 1$ and $0 \leqslant \varphi(x) \leqslant 1$. The function that we have just constructed is also called the *Uryson function*. ■

To apply this result, consider the extension of a bounded function from a closed subset of a normal space to the whole space. Note first that the major Uryson lemma is equivalent to the statement that there exists a continuous function $\varphi_{a,\,b}(x)$ that satisfies the following

$$\varphi_{a,\,b}|_A \equiv a,\ \varphi_{a,\,b}|_B \equiv b,\ a \leqslant \varphi_{a,\,b}(x) \leqslant b,\ x \in X,$$

where a, b $(a < b)$ are arbitrary real numbers. In fact, if $\varphi(x)$ is the Uryson function then the function $\varphi_{a,\,b}(x) = (b - a)\varphi(x) + a$ is the one required.

THEOREM 3 (TIETZE-URYSON). *For any bounded continuous function $\varphi : A \to R^1$ defined on a closed subset A of a normal space X, there exists a continuous function $\Phi : X \to R^1$ such that $\Phi|_A \equiv \varphi$ and* $\sup\limits_{(X)} |\Phi(x)| = \sup\limits_{(A)} |\varphi(x)|$.

PROOF. We shall construct the function Φ as the limit of a certain sequence of functions. Put $\varphi_0 = \varphi$ and

$$a_0 = \sup|\varphi(x)|, \quad A_0 = \left\{x : \varphi_0(x) \leqslant -\frac{a_0}{3}\right\},\ B_0 = \left\{x : \varphi_0(x) \geqslant \frac{a_0}{3}\right\}.$$

It is clear that the sets A_0, B_0 are closed and disjoint. By the major Uryson lemma, there exists a continuous function $g_0 : X \to R^1$ such that $|g_0(x)| \leqslant \dfrac{a_0}{3}$ and

$$g_0(x) = \begin{cases} -a_0/3 & \text{if } x \in A_0, \\ a_0/3 & \text{if } x \in B_0. \end{cases}$$

Now, we define the function φ_1 on A by the equality $\varphi_1 = \varphi_0 - g_0$. The function φ_1 is therefore continuous and $a_1 = \sup\limits_{(A)} |\varphi_1| \leqslant \dfrac{2}{3} a_0$. Similarly, by introducing the notation

$$A_1 = \left\{x : \varphi_1(x) \leqslant -\frac{a_1}{3}\right\}, B_1 = \left\{x : \varphi_1(x) \geqslant \frac{a_1}{3}\right\}$$

and choosing a Uryson function g_1 such that $|g_1(x)| \leqslant \frac{a_1}{3}$ and

$$g_1(x) = \begin{cases} -a_1/3 & \text{when} \quad x \in A, \\ a_1/3 & \text{when} \quad x \in B, \end{cases}$$

we put $\varphi_2 = \varphi_1 - g_1$ and $a_2 = \sup\limits_{(A)} |\varphi_2| \leqslant \frac{2}{3} a_1$ on the set A.

Thus, we are constructing a sequence of functions $\varphi_0 = \varphi, \varphi_1, \ldots, \varphi_n, \ldots,$ that are continuous on A, and a sequence of functions $g_0, g_1, \ldots, g_n, \ldots,$ that are continuous on X such that

$$\varphi_{n+1} = \varphi_n - g_n, \quad |g_n(x)| \leqslant a_n/3, \quad a_{n+1} \leqslant \frac{2}{3} a_n,$$

where $a_n = \sup\limits_{(A)} |\varphi_n(x)|$, $n = 0, 1, 2, \ldots$. Hence,

$$|\varphi_n(x)| \leqslant \left(\frac{2}{3}\right)^n a_0, \quad |g_n(x)| \leqslant \left(\frac{2}{3}\right)^n \frac{a_0}{3}.$$

The latter inequality means that the series $\sum\limits_{n=0}^{\infty} g_n(x)$ converges absolutely and uniformly to a continuous function in X. Having denoted its sum by $\Phi(x)$, we obtain an estimate

$$|\Phi(x)| \leqslant \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \frac{a_0}{3} = a_0.$$

Let $x \in A$, then the partial sum $S_n(x) = g_0(x) + \ldots + g_n(x)$ by the method of construction of the functions $\varphi_{n+1}(x)$ will equal $\varphi_0(x) - \varphi_n(x)$, and $\varphi_n(x) \to 0$. Therefore, $\Phi(x) = \varphi_0(x) = \varphi(x)$ for any $x \in A$. ■

The Tietze-Uryson theorem can be generalized for a mapping of the space X into an n-dimensional cube.

COROLLARY. *Any continuous mapping $\varphi : A \to I^n$ of a closed subset A of a normal space X into an n-dimensional cube I^n can be extended to a continuous mapping $\Phi : X \to I^n$.*

Exercise. Prove the corollary.

Hint: Use a coordinate system in R^n and apply the Tietze-Uryson theorem to the components of the mapping φ.

And now the following important fact can be proved.

THEOREM 4. *A metric space is normal.*

THE PROOF of this theorem is left to the reader.

13. COMPACT SPACES AND THEIR MAPPINGS

1. **The Notion of Compact Space.** We turn now to the study of a quite important class of topological spaces that are characterized by the properties of their open coverings. These properties are an abstract (and convenient) analogue of the property of compactness of a closed segment or an n-dimensional cube (or ball) known from analysis. Compact spaces and their mappings arise in many branches of mathematics.

We shall first discuss some of the ideas related to the coverings of topological spaces. Let $\sigma = \{A\}$ be some system of subsets A of a set X. The union of all A from σ will be denoted by $\tilde{\sigma}$ and called the *σ-field*.

We shall now extend the idea of a covering mentioned in Sec. 1 and given after the definition of a base for a topology.

DEFINITION 1. A family σ is called a *covering of a subspace* Y of a topological space X if $\tilde{\sigma} \supset Y$.

In particular, σ is a covering of the space X if $\tilde{\sigma} = X$, which agrees with the notion of covering used earlier in Sec. 1.

DEFINITION 2. A covering σ is said to be a *refinement* of a covering σ' ($\sigma > \sigma'$) if each element of σ is contained in some element of the system σ'. The refinement relation introduces a partial ordering on the set of all coverings of the space.

Coverings consisting of a finite (or countable) number of elements are said to be *finite* (or *countable*), respectively.

DEFINITION 3. A covering σ of a space X is said to be *locally finite* if each point $x \in X$ possesses a neighbourhood which intersects with only a finite number of elements of σ. Coverings consisting of open sets are particularly important and said to be *open*.

There are many important properties of spaces which are closely related to the properties of open coverings. Hence, the following classes of spaces are singled out.

DEFINITION 4. A Hausdorff topological space X is said to be (A_1) *compact*, (A_2) *finally compact*, (A_3) *paracompact* if an open covering that is (a_1) finite, (a_2) countable, and (a_3) locally finite, respectively, can be made a refinement of any of the open coverings of the space.

Exercise 1°. Verify that we will obtain an equivalent definition of (A_1) and (A_2) if we require that a covering of (a_1)- or (a_2)-type, respectively, could be selected from any open covering of the space.

It is important that the properties A_i, $i = 1, 2, 3$, of a topological space are inherited by each of its closed subsets considered as a subspace (prove it by yourself!).

EXAMPLES.

1. Let $X = [a, b] \subset R^1$ be a space endowed with the topology induced by that from R^1. The space X is compact, since, by the Heine-Borel theorem, a finite subcovering can be picked from any covering of X with intervals.
2. Let $X = R^1$; this is an example of a noncompact space, the reason being that a

finite covering cannot be picked from the covering $\{(-n, n)\}_{n=1}^{\infty}$. (However, a countable subcovering can be chosen from any open covering of R^1. Therefore R^1 is finally compact. Prove it.)

Similar reasoning demonstrates that the space R^n is also non-compact, nor are any of its unbounded subsets. Hence, it follows, in particular, that the requirement for a compact subset in R^n to be bounded is a necessary condition.

3. The space $X = R^1$ is paracompact. In fact, let $\{U_\alpha\}$ be an open covering of R^1. Then $R^1 = \bigcup_{n=-\infty}^{+\infty} [n, n+1]$. Each line-segment $[n, n+1]$ is 'a little' extended to the interval $(n - \varepsilon, n + 1 + \varepsilon)$. Consider the covering $\{U_\alpha \cap (n - \varepsilon, n + 1 + \varepsilon)\}$ of the line-segment $[n, n + 1]$. A finite covering $V_1^n, \ldots, V_k^n$ can be singled out of it. The union of such coverings (for all n) produces a locally finite covering of R^1 which is a refinement of $\{U_\alpha\}$. ♦

If $Y \subset X$ is a subspace of a Hausdorff space X then by considering coverings of the space Y which are open in the hereditary topology from X, we obtain, from Definition 4, the concept of compact, finally compact and paracompact subspace (a compact, finally compact and paracompact set Y in the space X is also often spoken about). We could, in an equivalent manner, consider the coverings of the space Y which are open in X. It is useful to note, moreover, that a closed set $Y \subset X$ inherits the properties A_i, $i = 1, 2, 3$, from the space X. In fact, to any open covering $\sigma = \{V_\alpha\}$ of the space Y, where $V_\alpha = Y \cap U_\alpha$, and U_α is open in X, there corresponds an open covering $\sigma_* = \{U_\alpha, U_* = X \setminus Y\}$ of the space X_1. Now, we pick a refinement $\tilde{\sigma} > \sigma_*$ (of a_i-type) of the space X. It is easy to reduce the covering $\tilde{\sigma}$ which we obtain to the covering $\tilde{\sigma}_Y$ of the subspace Y by intersecting the elements of $\tilde{\sigma}$ with Y and discarding those elements which are contained in V_*. It is obvious that $\tilde{\sigma}_Y > \sigma$.

The following theorem is often used in analysis.

THEOREM 1. *Any infinite set $Z \subset X$ of a compact space X has a limit point in X.*

PROOF. Assume, on the contrary, that $Z' = \varnothing$. Then $\bar{Z} = Z$. Therefore, Z is closed and, consequently, compact. On the other hand, each point $z \in Z$ is isolated in X. This implies that there exists an open neighbourhood $\Omega(z)$ in X such that $\Omega(z) \cap Z = z$. Neighbourhoods $U(z) = \Omega(z) \cap Z$ that are open in Z form an infinite covering of the space Z from which a finite subcovering cannot be selected, and so we arrive at a contradiction to the assumption that Z is compact. ■

The concept of compactness is intimately related to the concept of closedness. This is demonstrated by the following statement.

THEOREM 2. *If X is a compact subspace of a Hausdorff space Y, then X is closed.*

PROOF. Let $y \in Y \setminus X$. For any point $x \in X$, since Y is Hausdorff, there are open neighbourhoods $U_x(y)$, $U_y(x)$ of the points y, x such that $U_x(y) \cap U_y(x) = \varnothing$. The family $\{U_y(x)\}_{x \in X}$ forms a covering of X. Because X is compact, there is a finite subcovering $\{U_y(x_i)\}_{i=1}^{k}$. It is easy to see that the sets $U(X) = \bigcup_{i=1}^{k} U_y(x_i)$

and $\bigcap_{i=1}^{k} U_{x_i}(y) = U(y)$ are open and disjoint. Thus, we have shown that a compact set X and a point not in it can be separated in a Hausdorff space by the disjoint neighbourhoods $U(X)$ and $U(y)$. Hence, it follows that the complement $Y \setminus X$ is open, and therefore X is closed. ■

Exercise 2°. Prove that a compact space is regular.

DEFINITION 5. A system $\{M_\alpha\}$ of subsets of a space X is said to be *centred* if any of its finite subsystems possesses a nonempty intersection.

A dual statement of the definition of a compact space is the following theorem.

THEOREM 3. *A Hausdorff space X is compact if and only if any centred system of its closed subsets $\sigma = \{M_\alpha\}$ possesses a nonempty intersection.*

PROOF. Let $\sigma = \{M\}$ be an arbitrary centred system of closed subsets of the space X, and let X be compact. We show that $\bigcap_{M \in \sigma} M \neq \varnothing$. Assume the contrary, i.e., $\bigcap_{M \in \sigma} M = \varnothing$. Then $\bigcup_{M \in \sigma} (X \setminus M) = X$, i.e., the system $\{X \setminus M\}_{M \in \sigma}$ is an open covering of X. Since X is compact, there exists a finite subcovering $\{X \setminus M_k\}_{k=1}^{n}$. Therefore $\bigcup_{k=1}^{n} (X \setminus M_k) = X$ and, consequently, $\bigcap_{k=1}^{n} M_k = \varnothing$, which is contrary to the assumption that the system σ is centred.

Let the intersection $\bigcap_{M \in \sigma} M$ be nonempty for any centred system $\sigma = \{M\}$ of closed subsets. Let $\{U_\alpha\}$ be an arbitrary open covering of X. Then the system $\{X \setminus U_\alpha\}$ has the empty intersection and, by the assumption, is not centred.

Thus, the subsystem $\{X \setminus U_{\alpha_i}\}_{i=1}^{s}$ has the empty intersection for some $\alpha_1, \alpha_2, \ldots, \alpha_s$, whence $\{U_{\alpha_i}\}_{i=1}^{s}$ is a finite subcovering of the covering $\{U_\alpha\}$. Therefore, the space X is compact. ■

We now consider the property of paracompactness. It is interesting to examine the relation of paracompactness to the other properties of topological spaces. Consider the so-called locally compact spaces.

DEFINITION 6. A space X is said to be *locally compact* if it is Hausdorff and each point $x \in X$ possesses a neighbourhood $U(x)$ whose closure is compact.

One example of a locally compact space is the space R^n; another is a two-dimensional manifold (see Sec. 4, Ch. II).

THEOREM 4. *If a topological space X is locally compact then it is regular.*

PROOF. Let $a \in X$ be an arbitrary point, and $F \subset X$ a closed set not containing the point a. Then $X \setminus F$ is open, and $a \in X \setminus F$. Because the space X is locally compact,

there is an open neighbourhood $V(a)$ such that $\overline{V(a)}$ is compact, and $\overline{V(a)} \subset X \setminus F$. Therefore, $\overline{V(a)} \cap F = \varnothing$, and for any point $x \in F$, there exists an open neighbourhood $U(x)$ which is disjoint with $\overline{V(a)}$ (see the proof of Theorem 2, viz., the passage about the 'separation' of a compact set and a point in a Hausdorff space). Put $U(F) = \bigcup_{x \in F} U(x)$, then $V(a) \cap U(F) = \varnothing$. ■

The paracompact space of Example 3 is a special case of the spaces described by the following theorem.

THEOREM 5. *Let X be a locally compact space and $X = \bigcup_{n=1}^{\infty} C_n$, where C_n is a compact set, then X is paracompact.*

PROOF. We represent X first as a countable union of nested open sets whose closures are compact. We shall construct these sets by induction. First of all, we set $U_n = \varnothing$ when $n \leqslant 0$. We assume the neighbourhood of the set C_1 whose closure is compact to be U_1. Now that the set U_n is constructed, we take a neighbourhood with a compact closure of the set $\bar{U}_n \cup C_{n+1}$ and call it U_{n+1}. The existence of such neighbourhoods is stipulated by the local compactness of the space X.

Now, let $\{V_\alpha\}_{\alpha \in M}$ be an arbitrary open covering of X. Denote the compact set $\bar{U}_n \setminus U_{n-1}$ by D_n. The open set $U_{n+1} \setminus \bar{U}_{n-2}$ is thus a neighbourhood of the set D_n, and the family of sets

$$\{V_\alpha \cap (U_{n+1} \setminus \bar{U}_{n-2})\}_{\alpha \in M} = \{W_\alpha^n\}_{\alpha \in M}$$

forms an open covering of the set D_n. We select one of its finite subcoverings $\{W_m^n\}_{m=1}^{p_n}$. Having performed the described procedure for all n, we obtain a countable covering $\bigcup_{n=1}^{\infty} [\{W_m^n\}_{m=1}^{p_n}]$ (of the whole space X) which is a refinement of the covering $\{V_\alpha\}_{\alpha \in M}$.

We now show that this covering is locally finite. Let x be an arbitrary point from X, and $n_0 = \min\{n : x \in U_n\}$. Since $x \bar{\in} U_{n_0 - 1}$, there exists a neighbourhood $O(x)$ of x lying in U_{n_0} such that $O(x) \cap \bar{U}_{n_0 - 2} = \varnothing$. Consequently, $O(x)$ can only intersect the sets W_m^k, where $1 \leqslant m \leqslant p_k$, $n_0 - 2 \leqslant k \leqslant n_0 + 1$. By the method of construction, the number of such sets is finite. ■

COROLLARY. *If a locally compact space X possesses a countable base then it is paracompact.*

In fact, if a space is locally compact then it possesses a base $\{U^C\}$ of open sets such that $\bar{U}^C$ is compact. By choosing from the countable base those sets of which U^C consists, we obtain a countable base $\{V_i^C\}_{i=1}^{\infty}$ such that $\bar{V}_i^C$ is compact for any i. Then $X = \bigcup \bar{V}_i^C$, and its paracompactness follows from Theorem 4. ■

We now investigate the relationship between the concepts of compactness and normality.

THEOREM 6. *A compact space X is normal.*

PROOF. First establish that X is regular. Let $A \subset X$ be a closed subset, $x \in X \setminus A$. Since X is Hausdorff, for any $y \in A$, there exist the neighbourhoods $U_x(y)$, $U_y(x)$ of the points y, x such that $U_x(y) \cap U_y(x) = \varnothing$. The system $\{U_x(y)\}_{y \in A}$ forms a covering of A; since A is compact, a finite subcovering $\sigma = \{U_x(y_i)\}_{i=1}^{m}$ can be singled out. Because $U_x(y_i)$ is included in the closed set $X \setminus U_{y_i}(x)$, $\bar{U}_x(y_i) \subset X \setminus U_{y_i}(x) \subset X \setminus \{x\}$. We find that $\bigcup_{i=1}^{m} \bar{U}_x(y_i) \subset X \setminus \{x\}$.

But $\bigcup_{i=1}^{m} \bar{U}_x(y_i)$ is closed, therefore, $X \setminus \bigcup_{i=1}^{m} \bar{U}_x(y_i) = U_A(x)$ is an open neighbourhood of the point x. The union $\bigcup_{i=1}^{m} U_x(y_i) = V_x(A)$ is an open neighbourhood of the set A in X. It is evident that $V_x(A) \cap U_A(x) = \varnothing$, and the regularity of X has thus been proved.

We now have to prove that X is normal. Let sets A and B be closed in X, and $A \cap B = \varnothing$. Then for any point $x \in X$, there exists an open neighbourhood $U(x)$ for which at least one of the relations $\bar{U}(x) \cap A = \varnothing$, $\bar{U}(x) \cap B = \varnothing$ is true because X is regular. Consider the covering $\{U(x)\}_{x \in X}$ of the space X with such neighbourhoods and select a finite covering $\{U_{\alpha_i}\}_{i=1}^{m}$ from it. For each U_{α_i}, at least one of the relations $\bar{U}_{\alpha_i} \cap A = \varnothing$, $\bar{U}_{\alpha_i} \cap B = \varnothing$ is fulfilled. Let $U' = \cup U_{\alpha_i}$ be the union of those sets for which $\bar{U}_{\alpha_i} \cap A = \varnothing$, and similarly, $V' = \cup U_{\alpha_i}$, $\bar{U}_{\alpha_i} \cap B = \varnothing$. It is easy to see that the open sets $X \setminus U'$, $X \setminus V'$ contain A and B, respectively, and are disjoint. Thus, the normality of X has also been proved. ∎

2. Mappings of Compact Spaces. We shall study some important properties of continuous mappings of compact spaces.

THEOREM 7. *Let X, Y be topological spaces, X compact, Y Hausdorff, and $f: X \to Y$ a continuous mapping. Then the image $f(X)$ is a compact subspace in Y.*

PROOF. Consider an arbitrary open covering $\{V_\alpha\}$ of the space $Z = f(X)$. It is obvious that $f: X \to Z$ is also continuous, therefore, $\{f^{-1}(V_\alpha)\}$ is an open covering of X. We select a finite covering $\{f^{-1}(V_{\alpha_i})\}_{i=1}^{m}$ which exists due to the compactness of X. Then $\{V_{\alpha_i}\}_{i=1}^{m}$ is a finite open covering of Z. ∎

Exercise 3°. Show that a closed surface (see Sec. 4, Ch. II) is a compact topological space.

A stronger statement is given by the following theorem.

THEOREM 8. *Let $f: X \to Y$ be a continuous mapping, X compact, and Y Hausdorff. Then f is a closed mapping.*

PROOF. Remember that any closed subset of a compact space is compact. Let $M \subset X$ be an arbitrary closed (and therefore compact) subset in X. From Theorem 7, the set $f(M)$ is compact in Y and is therefore closed due to Theorem 2. ∎

We hence derive an important test of a homeomorphism.

THEOREM 9. *Let the previous theorem be fulfilled,and the mapping f bijective, then f is a homeomorphism.*

PROOF. Consider the inverse mapping $f^{-1} : Y \to X$. Show that it is continuous. Let $A \subset X$ be an arbitrary closed subset. Since f is a closed mapping, $f(A) = (f^{-1})^{-1}(A)$ is closed in Y, which implies the continuity of the mapping f^{-1}. ■

Many examples of compact spaces arise when constructing factor spaces.

EXAMPLE 4. Let X be a Hausdorff factor space of some compact space Y. Then X is compact since it is a continuous image (with respect to the projection) of a compact space.

Consider a continuous numeric"' function $f : X \to R^1$ on a compact space X. The following Weierstrass theorem which plays an important part in mathematical analysis is valid for it.

THEOREM 10. *Any continuous function $f : X \to R^1$ on a compact space X is bounded and attains its maximum (and minimum) value.*

PROOF. In view of Theorem 7, the set $f(x)$ is compact. By Theorem 2, any compact subspace in R^1 is closed. It has already been mentioned that any compact subspace in R^n is bounded. Therefore, $f(X)$ is bounded and closed. It is the boundedness of $f(X)$ that implies the boundedness of the function f. Since a closed set contains all of its limit points, $\sup_{x \in X} f(X) \in f(X)$, and $\inf_{x \in X} f(x) \in f(X)$. This completes the proof of the theorem. ■

3. Products of Compact Spaces.

In this section,the following important theorem will be proved.

THEOREM 11 (TIHONOV). *The topological product $X = \prod_{\alpha \in M} X_\alpha$ of any system $\{X_\alpha\}_{\alpha \in M}$ of compact spaces is compact.*

PROOF. The compactness criterion which we shall use is that any centred system of closed subsets has a nonempty intersection. Let $\{N^\lambda\} = \sigma_0$ be an arbitrary centred system of closed subsets in X. In the set of all such systems, consider the partial ordering relation determined by the inclusion relation $\sigma'' > \sigma'$ if any set from σ' is a subset of σ''. Let G be the set of all systems σ such that $\sigma > \sigma_0$. It is clear that any strictly ordered subset from G has a maximal element (the union). Then by Zorn's lemma, there is a maximal centred system $\bar{\sigma}$ in G, i.e., a system such that for any system $\sigma \in G$, either $\bar{\sigma} > \sigma$ or $\bar{\sigma}$ and σ are incomparable.

Let $\bar{\sigma} = \{N^\gamma\}$. It is easy to show that any finite intersection of elements from $\bar{\sigma}$ belongs to $\bar{\sigma}$, and also that any closed set M which intersects with any N^γ belongs to $\bar{\sigma}$. (Verify this property!) Clearly, if it is shown that $\bar{\sigma}$ possesses a nonempty intersection, i.e., $\bigcap_{N^\gamma \in \bar{\sigma}} N^\gamma \neq \varnothing$, then the proof will be completed. We denote the projection on the factor X_α by $\pi_\alpha : X \to X_\alpha$. For a certain α, the system $\{\pi_\alpha(N^\gamma)\}_{\bar{\sigma}} = \{N^\gamma_\alpha\} = \bar{\sigma}_\alpha$ is centred (the constituent sets need not be closed) in X_α, and therefore the system $\{\bar{N}^\gamma_\alpha\}$ is also centred. By taking into account that X_α is com-

pact, there exists an element $x_\alpha \in X_\alpha$ such that for any of its neighbourhoods $U_\alpha = U(x_\alpha)$, the intersection $U_\alpha \cap N^\gamma_\alpha \neq \varnothing$ for any $N^\gamma_\alpha \in \bar{\sigma}_\alpha$. Consider now an element $x = \{x_\alpha\} \in X$. Each of its neighbourhoods $U = U(x)$ contains the closure of a certain elementary neighbourhood of the form

$$(U_{\alpha_1} \times U_{\alpha_2} \times \ldots \times U_{\alpha_s} \times \prod_{\alpha \neq \alpha_1, \ldots, \alpha_s} X_\alpha) = U_{\alpha_1, \ldots, \alpha_s},$$

which is in turn the intersection of a finite number of neighbourhoods of the form $(U_{\alpha_i} \times \prod_{\alpha \neq \alpha_i} X_\alpha) = V_{\alpha_i} \subset X$. It is clear that V_{α_i} intersects all the sets $N^\gamma \in \bar{\sigma}$, since $U_{\alpha_i} \cap N^\gamma_{\alpha_i} \neq \varnothing$ for all γ. Consequently, $\bar{V}_{\alpha_i} \in \bar{\sigma}$ and therefore

$$\bar{U}_{\alpha_1, \ldots, \alpha_s} = \bigcap_{i=1}^{s} \bar{V}_{\alpha_i} \in \bar{\sigma}.$$

Hence, the neighbourhood $U = U(x)$ intersects all $N^\gamma \in \bar{\sigma}$. Since U is arbitrary, we deduce that $x \in \bigcap_{N^\gamma \in \bar{\sigma}} N^\gamma$ (and therefore $x \in \bigcap_{N^\gamma \in \sigma} N^\gamma$). ■

Here are some examples which demonstrate how the compactness of a space can quickly be determined by the Tihonov theorem.

EXAMPLES.

5. The cube $I^n = \underbrace{[0, 1] \times [0, 1] \times \ldots \times [0, 1]}_{n}$ is a compact space since it is the product of n line-segments.

6. The boundedness and closedness of a set in R^n are equivalent to its compactness. In fact, such a set in R^n can be included in a closed parallelepiped $[a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$, whose compactness can be established as in Example 5.

We have thus proved that the condition is sufficient. That the condition for closedness is necessary was proved in one of the theorems in this section, and that the condition for boundedness is necessary was noted in Example 2.

Exercises.

4°. Prove that the sphere S^n is compact.

5°. Verify that the n-dimensional torus $T^n = \underbrace{S^1 \times S^1 \times \ldots \times S^1}_{n}$ is compact.

EXAMPLES.

7. The projective space RP^n is compact because it is the factor space of the sphere S^n.

8. The lens space S^n/Z_p is compact for the same reason. ♦

4. Compactness in Metric Spaces. Compact metric spaces are often called *compacta*, and compact subspaces are called *compact sets of a metric space.*

The property of compactness in a metric space can be expressed in terms of convergent sequences.

DEFINITION 7. A set X of a metric space M is said to be *sequentially compact* if any sequence of its elements contains a subsequence which is convergent in M.

THEOREM 12. *A set X of a metric space M is compact if and only if it is closed and sequentially compact.*

PROOF. Let X be sequentially compact and closed. Then for any $\varepsilon > 0$, there exists a finite set of points $A_\varepsilon = \{x_k\}$ such that balls $D_\varepsilon(x_k)$ with centres in x_k and radius ε cover X*. In fact, otherwise, for some ε_0, there are points $x_1, x_2, \ldots, x_n, \ldots$ in X such that $\rho(x_n, x_{n+p}) \geqslant \varepsilon_0$ for all n, p. The availability of such a sequence is contrary to the sequential compactness of X. Thus, finite ε-nets exist for any $\varepsilon > 0$. Now, let $\{U\}$ be an arbitrary covering of X. Assuming that a finite subcovering cannot be singled out of it we find that in the finite ε_1-net A_{ε_1}, there is an element x_k such that the closed set $X \cap \bar{D}_{\varepsilon_1}(x_k) = X_1$ cannot be covered with any finite subsystem from $\{U\}$. It is easy to see that the set X_1 is closed and sequentially compact, and that its diameter is not greater than $2\varepsilon_1$. Applying the same reasoning to X_1, we can construct a set $X_2 \subset X_1$ with the same properties and a diameter not greater than $2\varepsilon_2 < 2\varepsilon_1$.

Thus, having taken into account the sequence $\varepsilon_n \to 0$, we can construct a system $\{X_n\}$ of closed, sequentially compact sets $X_{n+1} \subset X_n$ whose diameters tend to zero.

Exercise 6°. Show that $\bigcap\limits_{k=1}^{\infty} X_k \neq \varnothing$.

We infer from this that there exists a point $x_0 \in \bigcap\limits_{k=1}^{\infty} X_n$. Since $\{U\}$ is a covering, $x_0 \in U_\alpha$ for one of its elements U_α, and because U_α is open, there exists $\varepsilon > 0$ such that $D_\varepsilon(x_0) \subset U_\alpha$. By taking n sufficiently large for the diameter of X_n to be less than ε, we obtain the inclusion relations $X_n \subset D_\varepsilon(x_0) \subset U_\alpha$ and arrive at a contradiction with the assumption that X_n is uncoverable with a finite number of elements from $\{U\}$.

Closedness and sequential compactness follow from the closedness of a compact set (see Theorem 2) and existence of a limit point for any infinite sequence (see Theorem 1). ■

We leave the proof of the following useful statement to the reader.

THEOREM 13 (ON THE LEBESGUE NUMBER). *Let X be compact and $\{U\}$ an arbitrary open covering of X. Then there exists a real number $\delta > 0$ such that any set in X of diameter less than δ lies wholly in a certain element of the covering $\{U\}$.*

Exercise 7°. Let a metric space X be compact, and $f : X \to Y$ a continuous mapping. Prove that for any covering $U = \{U_\alpha\}$ of the space Y, there exists a Lebesgue number $\delta = \delta(U)$ such that for any subset A in X, of diameter less than δ, the image $f(A)$ is wholly contained in some element of the covering U.

One of the most important questions in mathematical analysis concerns the compactness of sets in function spaces. There are many special criteria of compactness in concrete spaces. One such criterion for the space $C_{[0,1]}$ which is widely used in mathematical analysis is given by the Arzelà theorem [46].

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* The set A_ε is called a *finite ε-net of X*.

14. COMPACTIFICATIONS OF TOPOLOGICAL SPACES. METRIZATION

1. **Compactifications.** The property of compactness proves to be quite useful and convenient in many questions. For this reason, it is natural to attempt to find a construction which would enable us to construct, for a given noncompact space, a compact space containing the given one and to investigate the relationships between the topologies, the properties of functions on these spaces, etc.

DEFINITION 1. A *compactification* of a topological space X is any compact space CX containing X as an everywhere dense subspace.

Consider a compactification which is used quite often, i.e., the Alexandrov *one-point compactification*.

DEFINITION 2. A compactification CX of the form $CX = X \cup \xi$, where ξ is a point not isolated in CX, is called a *one-point compactification* $X^\cdot$ of the space X.

THEOREM 1 (ALEXANDROV). *For a space X, there exists a one-point compactification $X^\cdot = X \cup \xi$ if and only if X is locally compact. Moreover, the topology of $X^\cdot$ coincides with the topology of X as a subspace in $X^\cdot$, and the topology on $X^\cdot$ is uniquely determined by the topology of X.*

PROOF. Let $X^\cdot = X \cup \xi$ be a compactification of X. We show that X is locally compact. In fact, it is evident that X is open in $X^\cdot$, therefore each point $x \in X$ possesses a neighbourhood $U(x)$ in the topology $X^\cdot$ such that $\bar{U}(x) \subset X$. However, $\bar{U}(x)$ is a closed set in $X^\cdot$ and therefore compact in $X^\cdot$ and X. The local compactness of X has thus been proved. Note also that any open set in $X^\cdot$ containing a point ξ is of the form $\xi \cup G$ where G is open in X. However, the closed set $X^\cdot \setminus (\xi \cup G) = X \setminus G$ is compact because $X^\cdot$ is compact.

Conversely, let X be locally compact. Let us describe a topology on $X^\cdot$ which satisfies the requirements of the theorem. We will consider all sets U which are open in X and also those of the form $\xi \cup G$, where G is an open set in X such that $X \setminus G$ is compact, to be open in $X^\cdot = X \cup \xi$. (The sets G exist due to the local compactness of X; it suffices to take the complement up to a compact closure of a certain neighbourhood.)

Exercise 1°. Verify that the family of sets described forms a topology on $X^\cdot$.

We have to show that the space $X^\cdot$ is compact with respect to the topology described above. First, we verify that $X^\cdot$ is Hausdorff. The separatedness of any two points from X follows from X being Hausdorff. We then show that ξ and $x \in X$ possess disjoint neighbourhoods. Having taken a neighbourhood $U(x)$ of the point x such that $\bar{U}(x)$ is compact in X, and setting $V(\xi) = \xi \cup (X \setminus \bar{U}(x))$, we have $U \cap V = \varnothing$. Thus, $X^\cdot$ is Hausdorff.

Now, let $\sigma = \{U_\alpha\}$ be an arbitrary open covering of $X^\cdot$. Then there exists an element U_{α_0} in σ that covers the point ξ and therefore $\xi \cup G = U_{\alpha_0}$, where $X \setminus G = K$ is compact in $X^\cdot$. The subcovering $\sigma' = \{U_\alpha\}_{\alpha \neq \alpha_0}$ covers the set K. Since K is compact, a finite subcovering can be extracted from σ'. We denote it by $\sigma'' = \{U_{\alpha_1}, \ldots, U_{\alpha_k}\}$. Then the collection $\{U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_k}\}$ is a finite covering of the space $X^\cdot$. Thus, the compactness of $X^\cdot$ has been proved. ■

A well-known example of a one-point compactification of the space R^n is the sphere S^n. One homeomorphism of R^n onto the punctured sphere S^n is, for instance, stereographic projection.

The Tihonov theorem given below singles out quite a wide class of spaces which are homeomorphic to a subset of a certain compact metric space called a Tihonov cube. Before we state it, we must give the necessary definitions.

DEFINITION 3. The product $\prod_{\alpha \in M} I_\alpha$ of line-segments, where M is a set of power $\tau, I_\alpha = I$, is called a *Tihonov cube* I^τ of weight τ.

For any τ, the product I^τ is compact. Note that if M is countable, then $I^\tau = I^\infty$ is the Hilbert cube (by definition).

DEFINITION 4. A space X is said to be *completely regular* if each closed subset and a point outside it are functionally separable.

DEFINITION 5. The *weight* $\omega(X)$ of a topological space X is a minimal cardinal number which is the power of some base (it being a set) for the topology on X.

THEOREM 2 (TIHONOV). *Any completely regular space X of weight τ is homeomorphic to a subset of the Tihonov cube I^τ.*

We now outline the proof of this theorem. Consider some set $\Sigma = \{\varphi_\alpha\}$ of continuous functions $\varphi_\alpha : X \to [0, 1]$ such that for any $x \in X$ and any open neighbourhood $U(x)$ of the point x, there is a function $\varphi_\alpha \in \Sigma$ for which $\varphi_\alpha(x) = 0, \varphi_\alpha|_{X \setminus U_\alpha} = 1$. These 'splitting' sets exist for completely regular spaces; e.g., the set of all continuous functions $f : X \to [0, 1]$. It happens that in a space of weight τ, such a family of power τ always exists.

A homeomorphism φ of the space X into a subset of the Tihonov cube is determined by associating each point x with a set of numbers $\varphi(x) = \{\varphi_\alpha(x)\}_\alpha$. Note that the compactness of $\overline{\varphi(X)}$ follows from the condition $\varphi(X) \subset I^\tau$ (due to the compactness of I^τ).

It is easy to see that any splitting set of functions on X determines a certain compactification of X. In fact, by identifying X with $\varphi(X)$, X itself can be considered to be embedded in $\overline{\varphi(X)}$ in its capacity as an everywhere dense subset. Consider the case when the splitting family $\{f_\alpha\}$ is maximal, i.e., coincides with the set of all continuous functions $\Phi = \{f : X \to [0, 1]\}$.

DEFINITION 6. The compactification $\hat{X}$ corresponding to the maximal family Φ is said to be *maximal* or *the Stone-Čech compactification* of the space X.

The Stone-Čech compactification possesses a number of useful properties. We list some of them.

THEOREM 3 (STONE-ČECH). *Each of the following three conditions is necessary and sufficient for a given compactification CX to be homeomorphic to the maximal extension $\hat{X}$:*

(1) *Each continuous function $f : X \to [0, 1]$ is extended to a continuous function $\bar{f} : CX \to [0, 1]$.*

(2) *Each continuous mapping $f : X \to B$ into a compact space B is continuously extended to $\bar{f} : CX \to B$.*

(3) *For any compactification $C'X$, there exists a continuous mapping $\varphi : CX \to C'X$ such that $\varphi|_X = 1_X$.*

2. Metrizability of Topological Spaces. We discuss here how to introduce a metric on a topological space so that it induces the same topology. Topological spaces which will admit such a metric are said to be *metrizable*. In particular, we may speak of the introduction of another metric on a metric space so that it generates the original topology but is itself more convenient, e.g., such that the space with this metric may be complete. Such metric spaces are said to be

topologically complete. One example of a topologically complete metric space is an interval $(a, b) \subset R^1$, $M = (-1, 1)$ in particular. Besides the standard metric $\rho(x, y) = |x - y|$ in which (M, ρ) is not complete, we can introduce a topologically equivalent metric

$$\rho_*(x, y) = \left| \frac{x}{\sqrt{1-x^2}} - \frac{y}{\sqrt{1-y^2}} \right|$$

induced by the homeomorphism of an interval and a straight line (the distance between the points of the interval is calculated in the metric ρ_* as the distance, in the usual metric, between their images on the straight line). It is easy to verify that ρ_* is a metric, (M, ρ) and (M, ρ_*) are homeomorphic, and that (M, ρ_*) is a complete metric space. One example of a topologically incomplete metric space is the set of rational numbers under the metric from R^1.

The Tihonov product of a countable number of metric spaces (M_n, ρ_n) is metrizable. In fact, if $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$ are elements from $\prod_{i=1}^{\infty} M_i$ then a metric can be given by the formula:

$$\bar{\rho}(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\rho_n(x_n, y_n)}{1 + \rho_n(x_n, y_n)}.$$

Exercise 2°. Verify that $\bar{\rho}$ is a metric and that the topology induced by it is equivalent to the Tihonov topology. In particular, the Hilbert cube I^∞ (i.e., a countable product of line-segments I) is a metrizable topological space.

It follows from the Tihonov compactification theorem that the following proposition, for instance, is true:

THEOREM 4 (URYSON). *A regular space with a countable base is metrizable.*

The proof is based on the regularity of the space and its countable base ensuring the normality of the space and hence its complete regularity.

According to the Tihonov theorem, such a space is embeddable in a metrizable space I^∞ and hence is itself metrizable.

In conclusion, we state A. Stone's important theorem, viz., that a *metrizable topological space is paracompact.*

FURTHER READING

This chapter is, basically, a recapitulation of the classical results of general topology which can be found in more detail in the extensive literature on the subject. We would recommend, first of all, some books marked by their systematic approach to the elements of general (i.e., set-theoretic) topology, e.g., *Introduction to Set Theory and General topology* [3] by Alexandrov, *General Topology* [45] by Kelley, and *General Topology* [6] by Alexandryan and Mirzachanyan. The book by Alexandrov contains a thorough account of set theory and many of the branches of general topology, illustrating them with a number of the classical examples of subspaces in R^1, R^2, R^3. We strongly recommend this book to students.

The basic topics of general topology can also be found in *Introduction to Dimension Theory* by Alexandrov and Pasynkov (subtitled *Introduction to Topological Space Theory*

and General Dimension Theory) [4], some parts of the treatise by Bourbaki *Topologie Générale* [18], Kuratowski's *Topology* [48], and Pontryagin's *Continuous Groups* [64].

The following books are useful extra material to the topics in this chapter: *First Course of General Topology in Problems and Exercises* [7] by Archangelsky and Ponomaryov, *Problems in Geometry* [61] by Novikov et al., and *Problems in Differential Geometry and Topology* [59] by Mishchenko et al.

As regards individual branches, we would make the following recommendations.

For the study of the concepts of topological and metric space and their continuous mappings (Secs. 1 and 2), the corresponding chapters of the above titles are recommended.

Sec. 3. The theory of factor space is most thoroughly expounded in the book by Kelley [45] (Ch. III), and that by Bourbaki [18] (Ch. I, Sec. 3).

Sec. 4. The classification of closed two-dimensional surfaces is presented well by Seifert and Threlfall [71] (Ch. VI), and a more modern approach by Bakelman et al. in *Introduction to Differential Geometry 'In the Large'* [13] (Sec. 10), and by Massey in *Algebraic Topology: An Introduction* [52], (Ch. I).

Sec. 5. The concepts of projective and lens spaces can be studied from Teleman [79] (Ch. I, Sec. 10), Seifert and Threllfall [71] (Chs. II, IX), and Fuchs et al. in *Homotopy Theory* [32] (Chs. I-II).

Secs. 6 and 7. The closure operations, boundary operator of a set in a topological space, etc., are referred to Alexandrov [3] (Ch. IV, Sec. 1), Alexandrov and Pasynkov [4] (Ch. I, Sec. 1), Bourbaki [18] (Ch. I, Sec. 1), and Kelley [45] (Ch. I).

Sec. 8. As far as continuous mappings are concerned, we would like to recommend the book by Kelley [45] (Ch. 3).

Secs. 9-12. The theory of product topology, connectedness, separation, normality of topological spaces is effectively handled by Alexandrov in [3] (Ch. VI, Sec. 4), Alexandrov and Pasynkov [4] (Ch. I, Sec. 8), Kelley [45] (Ch. 3), and Bourbaki [18] (Ch. I, Sec. 4).

Sec. 13. The concept of the compactness of a topological space can be referred to in many of the above books. We would recommend [3, 4, 18, 45, 67, 79]. Though somewhat differing in the terminology, we most closely follow [4]. In particular, it should be noted that at present no universally accepted term for the basic concept of compact space has been established. We have chosen the term 'compact spaces', whereas in [3, 4], they are said to be bicompact in accordance with the term introduced by Alexandrov and Uryson (see, for instance, *A Memoir on Compact Topological Spaces* [5] which can also be recommended for a more profound study of these questions).

Sec. 14. Somewhat more specialized aspects of general topology (i.e., compactification, metrization) are expounded in this section. For a deeper understanding, see [3, 4, 18, 79].

III

Homotopy Theory

One of the main methods of topology is to study the geometric properties of topological spaces algebraically. A number of approaches have been used in topology to associate a topological space with a number of algebraic objects, such as groups and rings. Algebraic topology has the same idea underlying it, viz., that there is a correspondence (or functor) associating a collection of topological spaces with a collection of certain algebraic objects, and continuous mappings of spaces with the corresponding homomorphisms. This functorial approach makes possible the reduction of a topological problem to a similar algebraic one. The solvability of the 'derived' algebraic problem in many cases implies that of the original topological problem.

One of the first concepts that have arisen on this way is that of the fundamental group of a topological space; later a more general concept, that of homotopy groups, has been introduced. It is to the latter that the present chapter is devoted.

1. MAPPING SPACES. HOMOTOPIES, RETRACTIONS, AND DEFORMATIONS

This section studies the set of all continuous mappings of one topological space to another. Various topologies can be introduced on this set thereby turning it into various topological spaces. The connectedness of this space is a particularly important question, which naturally leads us to the idea of homotopic mappings, wheas the consideration of special classes of mappings and their homotopies leads us to the concepts of the deformation of one space into another, of retraction, etc. All these concepts play important roles in homotopy theory.

1. The Space of Continuous Mappings. Consider the set $C(X, Y)$ of all continuous mappings from a topological space X to a topological space Y. The properties of this set, and many of those of the spaces X, Y, are interrelated. One simple example is that if X is one-point, then $C(X, Y) = Y$, where the sign '$=$' means a bijection.

A topology may be introduced on the set $C(X, Y)$, as on any other, in different ways. This makes us ask how it should be done in the most natural manner. An intuitive idea of the nearness of mappings may help considerably to settle the issue. Two mappings f_1, f_2 are said to be *near* if the images $f_1(x)$ and $f_2(x)$ for any point $x \in X$ are near in Y. If Y is a metric space then these notions are expressed in terms of the metric on Y. Hence, various topologies can be introduced on the set $C(X, Y)$, viz., the topology of pointwise convergence, the topology of uniform convergence, etc.

If (Y, ρ) is a metric space and X is compact, then the set $C(X, Y)$ is equipped with a metric μ thus:

$$\mu(f_1, f_2) = \sup_{x \in X} \rho(f_1(x), f_2(x)), f_1, f_2 \in C(X, Y).$$

DEFINITION 1. The topology τ_1 on $C(X, Y)$ as determined by the metric μ is called the *topology of uniform convergence.*

Exercises.

1°. Verify that μ possesses the properties of a metric.

2°. Consider a convergent sequence $f_n \xrightarrow{\mu} f$ in $C(X, Y)$ and give an equivalent definition of convergence in terms of a topology on Y. In the case of $X = [0, 1]$, compare this convergence with uniform convergence in $C_{[0, 1]}$.

DEFINITION 2. Consider in $C(X, Y)$ the sets $\{x_i, U_i\}_{i=1}^k = \{f \in C(X, Y) : f(x_i) \in U_i, i = 1, \dots, k\}$, where $x_1, x_2, \dots, x_k \in X$, $U_1, \dots, U_k$ are open sets in Y. The topology τ_2 generated by these sets in their capacity as a subbase is called the *topology of pointwise convergence* on $C(X, Y)$.

Exercises.

3°. Verify that sets of the form $\{x_i, U_i\}_i^k = 1$ and their finite intersections satisfy the criterion of a base.

4°. Consider a sequence $\{f_n\}$ that is convergent to f in a given topology and prove

that its convergence is equivalent to the convergence of the sequences $f_n(x) \to f(x)$ for any point $x \in X$.
5°. Given the set $\{Y_x\}_{x\in X}$ of the replicas of a space Y which have the elements $x \in X$ as their subscripts and the Tihonov product $\prod_{x\in X} Y_x$, show that the set $C(X, Y)$ can be identified with a subset of this product, and that the product topology induces the topology of pointwise convergence τ_2 on $C(X, Y)$.

The following definition supplies another version of a topology on the set $C(X, Y)$.

DEFINITION 3. Consider all possible sets of mappings of the form

$$[K, U] = \{f \in C(X, Y) : f(K) \subset U\},$$

where K is a compact set in X, and U an open set in Y. The topology τ_3 generated by these sets $[K, U]$ as a subbase is called the *compact-open topology* on $C(X, Y)$.

Exercises.
6°. Verify that the family of sets $[K, U]$, and their finite intersections satisfy the criterion of a base.
7°. Show that $\tau_2 < \tau_3$, and that $\tau_3 < \tau_1$ for a metric space.
8°. Prove that if Y is a metric space and X is compact, then the compact-open topology coincides with the topology of uniform convergence.
9°. If X is a noncompact space and Y is a metric space, then sequences of mappings which are uniformly convergent on any compact subset of X are often considered. Show that this convergence is equivalent to the convergence in the compact-open topology.
10°. Prove that if (Y, ρ) is a complete metric space, then the space $C(X, Y)$ is a complete metric space in the metric μ.
11°. Show that if X is locally compact, then the spaces $C(X \times Z, Y)$ and $C(Z, C(X, Y))$ are homeomorphic in the compact-open topology.

The space $C(X, Y)$ is often denoted by Y^X. The statement of Exercise 11 can then be written as $Y^{X\times Z} = (Y^X)^Z$ (the *exponential law*).

By way of an example, consider the space of ω-periodic continuous functions which are defined on the number line R^1. By virtue of their periodicity, each of these functions f is completely determined by its values on the line-segment $[0, \omega]$, and $f(0) = f(\omega)$. Therefore, we actually consider a set of functions on the line-segment $[0, \omega]$ whose extremities are 'glued together' or, what is equivalent, on the circumference S^1. This is the set $C(S^1, R^1)$ on which each of the topologies τ_1, τ_2, τ_3 may be introduced.

The function space can be considered in the same way on the torus:

$$T^n : C(T^n, R^1) = C(\underbrace{S^1 \times \dots \times S^1}_{n}, R^1).$$

This can be interpreted as the space of periodic functions in n variables.

2. Homotopy. It turns out in many problems that two mappings, one of which can be changed 'without abruptness', i.e., deformed into the other, are liable to possible identification. A continuous deformation of one mapping into another can be naturally thought of as a path in the space $C(X, Y)$ which begins and ends at

given points f_1 and f_2. Brouwer made the concept of continuous deformation more precise with the aid of the following concept of homotopy.

DEFINITION 4. Two continuous mappings $f_0, f_1 \in C(X, Y)$ are said to be *homotopic* ($f_0 \sim f_1$) if there exists a continuous mapping $f : X \times [0, 1] \to Y$ such that $f(x, 0) = f_0(x), f(x, 1) = f_1(x)$ for all $x \in X$.

The mapping f is often called a *homotopy* connecting the mapping f_0 to f_1.

Thus, if $f_0 \sim f_1$ then there is a family of mappings $f_t : X \to Y$ that depend on a numerical parameter $t \in [0, 1]$ and connect the mapping f_0 to f_1 so that the mapping $X \times [0, 1] \to Y$ induced by this family by the rule $(x, t) \to f_t(x)$ is continuous. The converse is obvious.

Exercise 12°. Show that giving a homotopy $f : X \times [0, 1] \to Y$ is equivalent to specifying a path s in $C(X, Y)$ (the topology being τ_3, X locally compact).

EXAMPLES.

1. Let $X = Y = R^n$, $f_0(x) = x$, $f_1(x) = 0$ for all $x \in R^n$. We define $F : R^n \times I \to R^n$ as $F(x, t) = (1 - t)x, t \in I$. It is easy to see that F is a homotopy between f_0 and f_1.
2. Let X be an arbitrary space, Y a convex subset in R^n, and $f_0, f_1 \in C(X, Y)$ arbitrary continuous mappings. Then the mapping $F : X \times I \to Y$, given by the formula $F(x, t) = tf_1(x) + (1 - t)f_0(x)$, is a homotopy between f_0 and f_1. ♦

Note that the concept of homotopy is related to the mapping extension problem. In fact, let $f, g : X \to Y$ be two continuous mappings. Defining a mapping $\varphi : X \times \{0\} \cup X \times \{1\} \to Y$ by the formulae $\varphi(x, 0) = f(x), \varphi(x, 1) = g(x)$, it is easy to see that $f \sim g$ if and only if there exists an extension of φ to $X \times [0, 1]$.

THEOREM 1. *A homotopy is an equivalence relation on the set* $C(X, Y)$.

PROOF. The reflexivity ($f \sim f$) is established by using the homotopy $F(x, t) \equiv \equiv f(x)$.

Symmetry. Assume that $f_0 \sim f_1$ with the homotopy $F(x, t)$, then $\bar{F}(x, t) = = F(x, 1 - t)$ defines a homotopy from f_1 to f_0, i.e. $f_1 \sim f_0$.

Transitivity. Let $f_0 \sim f_1, f_1 \sim f_2$ with homotopies $F_1(x, t), F_2(x, t)$, respectively. Then the mapping

$$H(x, t) = \begin{cases} F_1(x, 2t), & 0 \leqslant t \leqslant 1/2, \\ F_2(x, 2t - 1), & 1/2 \leqslant t \leqslant 1 \end{cases}$$

is continuous, since its restriction to each of the closed sets $X \times \left[0, \frac{1}{2}\right]$ and $X \times \left[\frac{1}{2}, 1\right]$ is continuous. It is easy to see that $H(x, t)$ is a homotopy between f_0 and f_1. ■

The equivalence classes of homotopic mappings are called *homotopy classes*. The factor set $C(X, Y)/R$ is denoted by $\pi(X, Y)$. It is easy to see that $\pi(X, Y)$ is the set of path components of the space $C(X, Y)$. The homotopy class of a mapping $f \in C(X, Y)$ is denoted by $[f]$.

DEFINITION 5. A mapping $f \in C(X, Y)$ is called a *homotopy equivalence* if there exists a mapping $g \in C(Y, X)$ such that $gf \sim 1_X, fg \sim 1_Y$.

DEFINITION 6. A space X is said to be *homotopy equivalent* to a space Y, or X and Y are to have the same *homotopy type* if there exists a homotopy equivalence on $C(X, Y)$.

The concept of homotopy equivalence is a useful 'coarsening' of that of homeomorphism of two spaces. In fact, if $f : X \to Y$ is a homeomorphism then, having put $g = f^{-1} : Y \to X$, we shall have $gf = 1_X, fg = 1_Y$, i.e., the condition for the homotopy equivalence of X and Y. In view of this, the mapping g in the definition of a homotopy equivalence is said to be *homotopy inverse* to f.

The simplest (nonempty) topological space is one-point. We shall now consider which spaces have the same homotopy type as a point.

DEFINITION 7. A space X is said to be *contractible* if the identity mapping $1_X : X \to X$ is homotopic to a constant mapping (i.e., the mapping of X into a point $x_0 \in X$). The homotopy between them is called a *contraction* of the space X (into a point x_0).

Exercise 13°. Prove that any two mappings of a space X into a contractible space Y are homotopic to one another.

THEOREM 2. *A space is contractible if and only if it has the same type as a point.*

PROOF. Let X be contractible, and $\Phi : X \times I \to X$ a contraction of X to a point $x_0 \in X$. Denote the one-point space consisting of the point x_0 by Q. Let $\varphi : X \to Q$ be a mapping into the point x_0, and $j : Q \to X$ an embedding. Then $\varphi j = 1_Q$, and Φ is a homotopy connecting 1_X with $j\varphi$. Thus, φ is a homotopy equivalence between X and Q. The proof of the converse is left to the reader. ■

Exercises.

14°. Prove that any convex subset in R^n (in particular, R^n itself) is contractible.

15°. Prove that the space $X \times Y$ is contractible if X and Y are contractible spaces.

3. **Extending Mappings.** We now consider the mapping extension problem. It can be formulated thus: Can a given mapping $f : A \to Y$ defined on a subspace A of a space X be extended to the whole space X, i.e., is there a mapping $\varphi : X \to Y$ such that its restriction $\varphi|_A : A \to Y$ coincides with the mapping f? Such a mapping φ is called the extension of the mapping f.

The solution of this problem has only been found for some special cases, and a complete extension theory has not yet been created. One example of a partial solution of this problem is the Tietze-Uryson theorem for normal spaces, which we proved in Sec. 12, Ch. II.

The following theorem establishes the connection between the mapping extension problem and the concept of homotopy.

THEOREM 3. *Let $\varphi: S^n \to Y$ be a continuous mapping of the unit sphere. Then the following two conditions are equivalent:*

(i) *the mapping φ is homotopic to the constant mapping;*

(ii) *the mapping φ can be extended to the whole ball $\overline{D}^{n+1} \subset R^{n+1}$*

PROOF. (*i*) ⇒ (*ii*): Let $f \to c$, where c is the constant mapping of S^n into a point $p \in Y$. Let $F : S^n \times I \to Y$ be a homotopy between f and c. We specify the exten-

sion f' of the mapping f to the ball $\bar{D}^{n+1}$ as follows:

$$f'(x) = \begin{cases} p, & 0 \leqslant \|x\| \leqslant 1/2, \\ F\left(\frac{x}{\|x\|}, 2 - 2\|x\|\right), & \frac{1}{2} \leqslant \|x\| \leqslant 1. \end{cases}$$

It is easy to see that $f'|_{S^n} = f$ and that f' is continuous since its restrictions to each of the closed sets

$$\{x \in \bar{D}^{n+1} : 0 \leqslant \|x\| \leqslant 1/2\}, \{x \in \bar{D}^{n+1} : 1/2 \leqslant \|x\| \leqslant 1\}$$

are continuous.

(*ii*) $\Rightarrow$ (*i*). Let f', the extension of f to the whole ball $\bar{D}^{n+1}$, be given. Let $y_0 \in S^n$. We define the mapping Φ: $S^n \times I \to Y$ as

$$\Phi(x, t) = f'[(1 - t)x + ty_0].$$

It is clear that $\Phi(x, 0) = f'(x) = f(x)$, $\Phi(x, 1) = f'(y_0) = p \in Y$, and therefore $\Phi(x, t)$ is the required homotopy. ■

Exercises.

16°. Show that any mapping f of a space X to a contractible space Y is homotopic to a constant mapping (cf. Ex. 13).

17°. Using the result of the previous exercise, deduce from Theorem 3 that any mapping of the sphere S^n to a contractible space can be extended to the whole ball $\bar{D}^{n+1}$

4. **Retraction.** A special case of the extension problem is that of a retraction enunciated in the following manner.

DEFINITION 8. Let A be a subspace of X,and $1_A : A \to A$ the identity mapping. If there exists a mapping $r : X \to A$ such that $r|_A = 1_A$, then it is called a *retraction* of X onto A, and the space A a *retract* of X.

Exercises.

18°. Verify that any point of a topological space X is a retract of X.

19°. Verify that any linear subspace in R^n is a retract of R^n.

20°. If $Z = X \times Y$ is the Tihonov product of spaces and $p \in X$, $q \in Y$ are fixed points, then $A = X \times q$, $B = p \times Y$ are retracts of the space $X \times Y$, and the mappings $r_X : (x, y) \to (x, q)$, $r_Y : (x, y) \to (p, y)$ are the corresponding retractions.

21°. Show that the zero-dimensional sphere $S^0 = \{-1, 1\}$ is not a retract of the one-dimensional disc $\bar{D}^1 = [-1, 1]$.

Hint: Use the properties of connected spaces.

DEFINITION 9. If there exists a mapping $r : X \to A$ such that $r|_A \simeq 1_A$, then A is called a *weak retract* of X, whereas r a *weak retraction* of X on A.

It is easy to see that a retract is always a weak retract. Generally speaking, the converse is not correct, which is demonstrated by the following exercise.

Exercise 22°. Given a square $I^2 = [0, 1] \times [0, 1]$ and its subset A, a 'comb space' consisting of (a) vertical line-segments whose bases are at the points $(1/n, 0)$,

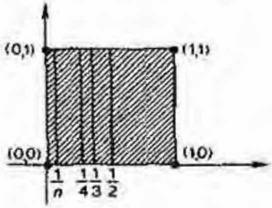

Fig. 54

$n = 1, 2, \ldots$; (b) (0, 0), and (c) the base of the square (Fig. 54). Show that (i) the set A is not a retract of the square I^2, (ii) A is a weak retract of I^2, (iii) if a finite number of teeth are left in the 'comb space' A, then the set A' that we obtain is a retract of I^2.

DEFINITION 10. A homotopy $D : X \times I \to X$ such that $D(x, 0) = x$, and $D(x, 1) \in A$ for all $x \in X$, is called a *deformation* of a space X into a subspace A.

DEFINITION 11. *If there exists a deformation of* X into A, $D : X \times I \to X$, such that $D(x, t) = x$ for $x \in A$, $t \in I$, then A is called a *strong deformation retract of* X, and D a *strong deformation retraction*.

EXAMPLE 3. A point is a strong deformation retract of any convex subset of R^n containing it.

Other examples of strong deformation retracts are given in the following exercises.

Exercises.

23°. Let a space X be contractible to a point $x_0 \in X$. Show that $x_0 \times Y$ is a strong deformation retract of the product $X \times Y$. In particular, consider a two-dimensional cylinder and show that its base is a strong deformation retract.

24°. Verify that the vertex of a cone in three-dimensional space is a strong deformation retract of the cone.

25°. Show that a strong deformation retract A of a space X is homotopy equivalent to X.

Hint: the embedding $i : A \to X$ and the retraction $D(1, x)$ of the space X onto A are homotopy inverse.

5. **Mapping Cylinder.** Consider some operations over topological spaces first.

The *topological sum (disjoint union)* $X \vee Y$ of two spaces X, Y is defined as the union of the disjoint replicas of X and Y.

The topology on $X \vee Y$ is defined as follows: V is open in $X \vee Y$ if and only if $V \cap X$ and $V \cap Y$ are open in X and Y, respectively.

If $f: A \to Y$ is a continuous mapping, where $A \subset X$, then X and Y can be glued together with respect to the mapping f. With this in mind, we introduce an equivalence relation on $X \vee Y$ thus:

$$R : x \sim y \text{ if } x \in A,\ y \in Y \text{ and } f(x) = y;$$
$$x_1 \sim x_2 \text{ if } x_1, x_2 \in A \text{ and } f(x_1) = f(x_2).$$

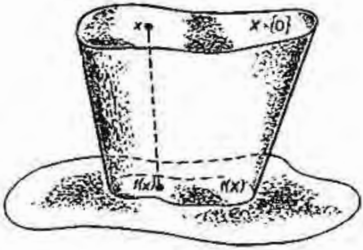

Fig. 55

The factor space of the space $X \vee Y$ with respect to the equivalence R is denoted by $X \cup_f Y$ and called the *sewing* of the spaces X and Y with respect to the mapping f. If, in particular, A is a point $x_0 \in X$, and the mapping $f: X \to Y$ carries x_0 into $y_0 = f(x_0)$, then the sewing $X \cup_f Y$ is called the *wedge* of the spaces X, Y and denoted by $X_{x_0} \vee_{y_0} Y$. It is easy to see that this is the factor space of the disjoint union $X \vee Y$ with respect to the equivalence relation gluing together the points $x_0 \in X$ and $y_0 \in Y$.

Exercises.

26°. Show that the homotopy type of the wedge $X_{x_0} \vee_{y_0} Y$ coincides with the homotopy type of the space Y if X is contractible to the point $x_0 \in X$.

27°. Prove that the line-segment $I = [0, 1]$ and the wedge $I_0 \vee_{p_0} S^1$, where $p_0 \in S^1$, $0 \in I$, have two different homotopy types.

DEFINITION 12. Let $f: X \to Y$ be a continuous mapping. Then we may assume that the mapping φ: $X \times \{1\} \to Y$, $\varphi(x, 1) = f(x)$ is defined, where $X \times \{1\}$ is a subspace of $X \times I$. The *cylinder Z_f of the mapping $f: X \to Y$* is a sewing $(X \times I) \cup_\varphi Y$ of the spaces $X \times I$ and Y with respect to the mapping φ.

The mapping cylinder can be represented as it is depicted in Fig. 55.

The notion of mapping cylinder is important seeing that X and Y can be considered to be subspaces of Z_f. Thus, the mapping f is replaced, in a sense, by the embedding of X into Z_f. Note also that Y is a strong deformation retract of Z_f, and the embedding of Y into Z_f is a homotopy equivalence (verify!).

DEFINITION 13. The cylinder of a constant mapping $c: X \to p$ is called a *cone* over the space X and denoted by CX.

THEOREM 4. *A mapping $f: X \to Y$ is homotopic to a constant one if and only if there exists an extension $\tilde{f}: CX \to Y$ of the mapping f.*

PROOF. If f is homotopic to the constant mapping $c_0: X \to (*)$, then

$$F: X \times I \to Y, F(x, 0) = x, F(x, 1) = (*).$$

Thus, F is constant on the upper base of the cylinder $X \times I$, and therefore induces the mapping F of the factor space $(X \times I)/R$, where R stands for shrinking the

upper base to a point. But the space $(X \times I)/R$ is homeomorphic to CX (verify!).

The proof of the converse statement is left to the reader. ■

Exercises.

28°. Let a mapping $f : A \to Y$ be continuous, $A \subset X$ closed in X and X, Y normal spaces. Prove that $X \cup_f Y$ is normal.

29°. Prove that $f : A \to Y$ can be extended to all $X (A \subset X)$ if and only if Y is a retract of $X \cup_f Y$.

2. CATEGORY, FUNCTOR AND ALGEBRAIZATION OF TOPOLOGICAL PROBLEMS

A description of a mathematical object in terms of categories implies that this object, for instance, a group or a space, is considered as a member of a collection of similar objects rather than separately. Intuitively, a category can be represented as a collection of sets (possibly, with an additional structure) and mappings agreeing with this structure. Correspondences between elements of different categories obeying special rules are called *functors*.

1. Category. DEFINITION 1. A *category* $\mathscr{A}$ is said to be *given* if there are given: (i) a certain collection of objects; (ii) for each ordered pair of objects X, Y, the set $\mathrm{Mor}_{\mathscr{A}}(X, Y)$ of *morphisms** from X to Y, and (iii) a mapping associating any ordered set of three objects X, Y, Z and any pair of morphisms $f \in \mathrm{Mor}_{\mathscr{A}}(X, Y)$, $g \in \mathrm{Mor}_{\mathscr{A}}(Y, Z)$ with their composition $gf \in \mathrm{Mor}_{\mathscr{A}}(X, Z)$. Thus, a commutative diagram of the morphisms in the given category (the morphisms being denoted by arrows) results:

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "gf"'] & & Z \end{tikzcd}$$

Furthermore, two properties must be fulfilled:

(A) *Associativity. If*

$$f \in \mathrm{Mor}_{\mathscr{A}}(X, Y),\ g \in \mathrm{Mor}_{\mathscr{A}}(Y, Z),\ h \in \mathrm{Mor}_{\mathscr{A}}(Z, W),$$

then

$$h(gf) = (hg)f \text{ in } \mathrm{Mor}_{\mathscr{B}}(X, W).$$

(B) *The existence of the identity element. For any object Y in* $\mathrm{Mor}_{\mathscr{A}}(Y, Y)$, *there exists a morphism 1_Y such that for any $f \in \mathrm{Mor}_{\mathscr{A}}(X, Y)$, $g \in \mathrm{Mor}_{\mathscr{A}}(Y, Z)$,*

$$1_Y f = f \text{ and } g 1_Y = g.$$

---

* It is assumed that $\mathrm{Mor}_{\mathscr{A}}(X, Y) \cap \mathrm{Mor}_{\mathscr{A}}(X', Y') = \phi$ when $X \neq X'$ and $Y \neq Y'$.

Note that the uniqueness of the element 1_Y follows from the above properties; this element is called the *identity morphism* of the object Y. If for two morphisms $f \in \mathrm{Mor}_{\mathscr{A}}(X, Y)$, $g \in \mathrm{Mor}_{\mathscr{A}}(Y, X)$, the equality $gf = 1_X$ is valid, then the morphism g is said to be *left inverse* of f, and f *right inverse* of g. A morphism which is both right and left inverse of f is said to be *two-sided inverse* of f.

DEFINITION 2. A morphism $f \in \mathrm{Mor}_{\mathscr{A}}(X, Y)$ is called an *equivalence* ($f : X \approx Y$) if there exists a morphism $f^{-1} \in \mathrm{Mor}_{\mathscr{A}}(Y, X)$ which is a two-sided inverse of f.

Exercise 1°. Prove that if a morphism $f \in \mathrm{Mor}_{\mathscr{A}}(X, Y)$ possesses a left inverse and a right inverse, then they coincide.

It follows from the exercise that if $f : X \approx Y$, then $f^{-1} : Y \approx X$.

Here are some important examples of categories.

1. The collection of sets and their mappings.
2. The collection of metric spaces and their continuous mappings.
3. The collection of topological spaces and their continuous mappings.
4. The collection of linear spaces and their linear mappings.
5. The collection of groups and their homomorphisms.
6. The collection of pairs of topological spaces and their continuous mappings.

By a *pair of topological spaces* (X, A), we mean a space X and its subspace A. The *mapping of pairs* $f : (X, A) \to (Y, B)$ is a mapping $f : X \to Y$ such that $f(A) \subset B$. ♦

Exercises.

2°. Show that in categories of Examples 2, 3, and 6, the homeomorphisms and only they are equivalences.

3°. Verify that the equivalences in the category of Example 1 are bijective mappings of sets.

4°. Show that the equivalences in Examples 4 and 5 are isomorphisms of linear spaces and groups, respectively.

2. FUNCTORS. We will consider natural mappings of one category to another, i.e., the mappings which preserve the identity elements and compositions of morphisms. Here we enunciate this concept more precisely.

DEFINITION 3. Let $\mathscr{A}$ and $\mathscr{B}$ be two categories. A *covariant functor* T from $\mathscr{A}$ to $\mathscr{B}$ is a mapping which associates each object X from $\mathscr{A}$ with an object $T(X)$ from $\mathscr{B}$, and assigns to each morphism $f : X_1 \to X_2$ in $\mathscr{A}$ a morphism $T(f) : T(X_1) \to T(X_2)$ in $\mathscr{B}$, while the following relations are true:

$$(1)\ T(1_X) = 1_{T(X)},\ (2)\ T(gf) = T(g)T(f).$$

Properties (1) and (2) of a functor can be visually represented as follows: any commutative diagram of the category $\mathscr{A}$ is mapped by a functor into the corresponding commutative diagram of the category $\mathscr{B}$:

$$\begin{tikzcd}
 & X_2 \arrow[dr, "g"] & & & & T(X_2) \arrow[dr, "T(g)"] & \\
X_1 \arrow[ur, "f"] \arrow[rr, "gf"'] & & X_3 & \arrow[r, dashed, "T"] & T(X_1) \arrow[ur, "T(f)"] \arrow[rr, "T(gf)"'] & & T(X_3)
\end{tikzcd}$$

EXAMPLE 7. A covariant functor is a correspondence associating a topological space with the set of all points that make it up, and a continuous mapping of spaces with a mapping of sets. This is a functor from the category of Example 3 to the category of Example 1. It is said to be *forgetful*, since it 'forgets' the topological space structure.

Similarly, a covariant functor from the category of metric spaces to the category of topological spaces with the topology induced by the metric 'forgets' the metric.

DEFINITION 4. A *contravariant functor* T from a category $\mathscr{A}$ to a category $\mathscr{B}$ is a mapping which associates each object X from $\mathscr{A}$ with an object $T(X)$ from $\mathscr{B}$, and each morphism $f : X_1 \to X_2$ with a morphism $T(f) : T(X_2) \to T(X_1)$ from $\mathrm{Mor}_{\mathscr{B}}(T(X_2), T(X_1))$, while the following relations are fulfilled:

$$(1)\ T(1_X) = 1_{T(X)},\ (2)\ T(gf) = T(f)T(g).$$

In other words, a contravariant functor transforms the commutative diagram of a category $\mathscr{A}$ into the commutative diagram of a category $\mathscr{B}$, reversing the arrows:

$$\begin{tikzcd} & X_2 \arrow[dr, "g"] & & & & T(X_2) \arrow[dl, "T(f)"'] & \\ X_1 \arrow[ur, "f"] \arrow[rr, "gf"'] & & X_3 \arrow[r, dashed, "T"] & & T(X_1) \arrow[rr, "T(gf)"'] & & T(X_3) \arrow[ul, "T(g)"'] \end{tikzcd}$$

Important examples of the functors studied in algebraic topology are homology group and homotopy group functors. These are functors from the category of topological spaces to the category of groups. In the next section, we shall dwell at length on homotopy group functors, whereas homology group functors will be considered in Ch. V.

We now consider an example of how a functor to the category of groups is applied to the investigation of some topological problems. In the previous section, the mapping extension problem was enunciated. We now formulate it as follows: let $A \subset X$ be a subspace of a topological space X, $i : A \to X$ the natural mapping associating any point $a \in A$ with itself, but in the space X (i.e., i is an embedding mapping), and $\varphi : A \to Y$ a mapping of the space A to a space Y. The mapping $\bar{\varphi} : X \to Y$ extends the mapping φ if and only if the diagram

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "\varphi"'] & & X \arrow[dl, "\bar{\varphi}"] \\ & Y & \end{tikzcd}$$

is commutative.

By means of a functor T (for example, covariant), we can derive an algebraic problem, viz., is there a homomorphism $T(\bar{\varphi})$ such that the diagram

$$\begin{tikzcd} T(A) \arrow[rr, "T(i)"] \arrow[dr, "T(\varphi)"'] & & T(X) \arrow[dl, "T(\bar{\varphi})"] \\ & T(Y) & \end{tikzcd}$$

is commutative?

It is clear that the solvability of the original problem entails that of our algebraic problem. Thus, the existence of the homomorphism $T(\bar{\varphi})$ is a necessary condition for an extension $\bar{\varphi}$ of the mapping φ to exist. For example, if the homomorphism $T(i)$ happens to be zero, and $T(\varphi)$ nonzero, then the homomorphism $T(\bar{\varphi})$ does not exist (otherwise, commutativity of the diagram would be violated), and then there is no extension $\bar{\varphi}$ of the mapping φ.

3. FUNCTORS OF HOMOTOPY GROUPS

In this section, we shall retrace our steps to the study of topics touching upon mapping spaces. In some cases, the set $\pi(X, Y)$ turns out to be a group, sometimes Abelian, and may be helpful in constructing various algebraic functors on the category of topological spaces and their continuous mappings. The construction and use of these functors form the basis for homotopy theory.

1. **The Homotopy Group of a Space.** Note at first that to each topological space Y and continuous mapping $f : X_1 \to X_2$ of topological spaces X_1, X_2, there corresponds the natural mapping

$$\pi^Y(f) : \pi(X_2, Y) \to \pi(X_1, Y).$$

More exactly, if $[\varphi] \in \pi(X_2, Y)$ then there is a unique element $[\varphi f]$ corresponding to $[\varphi]$ in $\pi(X_1, Y)$. Similarly, to any topological space X and continuous mapping $g : Y_1 \to Y_2$, there corresponds the mapping

$$\pi_X(g) : \pi(X, Y_1) \to \pi(X, Y_2).$$

Exercises.

1°. Describe the structure of $\pi_X(g)$ and prove the correctness of the definitions of $\pi^Y(f)$ and $\pi_X(g)$.

2°. Using the notes given, show that for a fixed Y, the correspondence $X \to \pi(X, Y)$ is a contravariant functor into the category of sets, and the correspondence $Y \to \pi(X, Y)$ (for a fixed X) is a covariant functor.

The correspondence $(X, Y) \to \pi(X, Y)$ is said to define a *bifunctor* from the category of topological spaces to the category of sets, which is covariant with respect to the second argument and contravariant with respect to the first.

A bifunctor π on the category of pairs of topological spaces determined by the correspondence $(X, A; Y, B) \to \pi(X, A; Y, B)$ may be considered in a similar way. Note that the homotopy $F(x, t)$ between mappings f and $g : (X, A) \to (Y, B)$ of pairs of spaces is understood to be a mapping of pairs $F : (X \times I, A \times I) \to (Y, B)$ such that

$$F(x, 0) = f(x), F(x, 1) = g(x).$$

Exercise 3°. Describe the structure of the mapping

$$\pi_{(X, A)}(f) : \pi(X, A; Y_1, B_1) \to \pi(X, A; Y_2, B_2)$$

naturally induced by a continuous mapping of pairs $f : (Y_1, B_1) \to (Y_2, B_2)$, and verify that the correspondence $(Y, B) \to \pi(X, A; Y, B)$ is a covariant functor.

DEFINITION 1. The pair (X, x_0) is called a *base point space*, the base point being $x_0 \in X$.

Now, we fix the pair $(I^n, \partial I^n)$, where I^n is an n-dimensional cube, and ∂I^n its boundary, and associate the pair (X, x_0) with the set $\pi(I^n, \partial I^n; X, x_0)$.

Remember that the elements $\pi(I^n, \partial I^n; X, x_0)$ are classes of the mappings of pairs $\varphi: (I^n, \partial I^n) \to (X, x_0)$ which are homotopic to one another and often called *spheroids*. Each of these mappings carries I^n to X, and ∂I^n to the point x_0. In addition, this property should be preserved when the mapping φ is changed in the course of the homotopy. The sets $\pi(I^n, \partial I^n; X, x_0)$ and $\pi(S^n, p_0; X, x_0)$ coincide (correspond bijectively). Here p_0 is a base point of the sphere S^n. In fact, we noted earlier that the factor space $I^n/\partial I^n$ is homeomorphic to the sphere S^n, the interior $\operatorname{Int} I^n$ of the cube I^n bijectively corresponding under this homeomorphism θ to the set $S^n \setminus p_0$, and the boundary ∂I^n being transformed into the point p_0 of the sphere S^n. A *relative homeomorphism* is then said to be given, viz.

$$\theta: (I^n, \partial I^n) \to (S^n, p_0).$$

Hence, to any mapping $f: (S^n, p_0) \to (X, x_0)$, there corresponds the mapping $f\theta: (I^n, \partial I^n) \to (X, x_0)$, and vice versa, to a mapping $g: (I^n, \partial I^n) \to (X, x_0)$, there corresponds the mapping $\bar{g}: (S^n, p_0) \to (X, x_0)$ which coincides with $g\theta^{-1}$ on $S^n \setminus p_0$, and carries the point p_0 into x_0.

Exercise 4°. Show that this correspondence between mappings ensures bijection between $\pi(S^n, p_0; X, x_0)$ and $\pi(I^n, \partial I^n; X, x_0)$.

Thus, we have given another interpretation of the set $\pi(I^n, \partial I^n; X, x_0)$, which makes it possible to consider the case when $n = 0$.

Exercise 5°. Show that the set $\pi(S^0, p_0; X, x_0)$ is the set of path components of the space X.

Consequently, we have defined a covariant functor $(X, x_0) \to \pi(I^n, \partial I^n; X, x_0)$ from the category of base point spaces to the category of sets.

The structure of the set $\pi(I^n, \partial I^n; X, x_0)$ is of the greatest interest as far as homotopy theory is concerned.

THEOREM 1. *The set $\pi(I^n; \partial I^n; X, x_0)$, $n > 1$, is an Abelian group. This group is called an n-dimensional homotopy group of the space X with the base point $x_0 \in X$ and is denoted by $\pi_n(X, x_0)$.*

PROOF. Let $[\varphi], [\psi] \in \pi(I^n, \partial I^n, X, x_0)$. We define the sum $[\varphi] + [\psi]$ as $[\varphi] + [\psi] = [\varphi + \psi]$, where the mapping $\varphi + \psi$ is defined thus: let

$$t = (t_1, t_2, \dots, t_n) \in I^n,\ t_i \in I = [0, 1],\ i = 1, \dots, n,$$

then

$$(\varphi + \psi)(t) = \begin{cases} \varphi(2t_1, t_2, \dots, t_n) & \text{when } 0 \leqslant t_1 \leqslant 1/2, \\ \psi(2t_1 - 1, t_2, \dots, t_n) & \text{when } 1/2 \leqslant t_1 \leqslant 1. \end{cases}$$

This definition can be illustrated visually by Fig. 56, where the square represents the face (t_1, t_2) of the cube I^n.

We define the zero element as the class of the constant mapping $\theta: (I^n, \partial I^n) \to (X, x_0)$ for which $\theta(I^n) = x_0$, and show that $[\varphi] + [\theta] = [\varphi]$ for any $[\varphi]$, i.e., $\varphi + \theta$

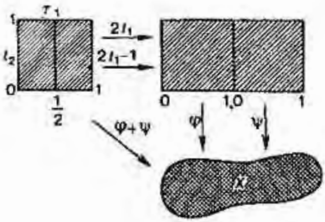

Fig. 56

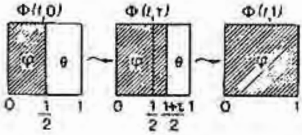

Fig. 57

is homotopic to φ. In fact, the required homotopy is determined by the mapping

$$\Phi : (I^n \times I, \partial I^n \times I) \to (X, x_0),$$

where

$$\Phi(t, \tau) = \begin{cases} \varphi\left(\dfrac{2t_1}{\tau + 1}, t_2, \dots, t_n\right) \text{ when } & 0 \leqslant t_1 \leqslant \dfrac{\tau + 1}{2}, \\ \quad x_0 & \text{when } \dfrac{\tau + 1}{2} \leqslant t_1 \leqslant 1, \tau \in I. \end{cases}$$

The homotopy $\Phi(t, \tau)$ is represented schematically in Fig. 57.

Exercises.

6°. Verify that the equality $[\theta] + [\varphi] = [\varphi]$ is also valid.

7°. Explain the reason why the equality $[\psi] + [\varphi] = [\varphi]$, when $[\psi] \neq [\theta]$, cannot be proved in the same way as the last two statements.

For any $[\varphi]$, the inverse element in $\pi_n(X, x_0)$ is the class $[\varphi\,\eta]$, where $\eta : I^n \to I^n$ is defined by the formula $\eta(t) = (1 - t_1, t_2, \dots, t_n)$. Thus, $(\varphi\eta)(t) = \varphi(1 - t_1, t_2, \dots, t_n)$.

To verify that $[\varphi] + [\varphi\eta] = [\theta]$, we show that a homotopy between the mappings $\varphi + \varphi\eta$ and θ is given by the mapping

$$\Phi(t, \tau) = \begin{cases} x_0 & 0 \leqslant t_1 \leqslant \tau/2, \\ \varphi(2t_1 - \tau, t_2, \dots, t_n, & \tau/2 \leqslant t_1 \leqslant 1/2, \\ \varphi(-2t_1 + 2 - \tau, t_2, \dots, t_n), & 1/2 \leqslant t_1 \leqslant 1 - \tau/2, \\ x_0, & 1 - \tau/2 \leqslant t_1 \leqslant 1. \end{cases}$$

In Fig. 58, this homotopy is represented diagrammatically.

Exercise 8°. Verify that the conditions for the homotopy ot pairs are fulfilled.

Finally, we have to verify the associativity of addition in $\pi_n(X, x_0)$ and the commutativity of addition when $n > 1$. We first prove the associativity.

Let $[\varphi], [\psi], [\mu] \in \pi_n(X, x_0)$. We shall show that

$$([\varphi] + [\psi]) + [\mu] = [\varphi] + ([\psi] + [\mu]).$$

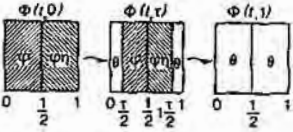

Fig. 58

Fig 59

It is easy to verify that the required homotopy is given by the mapping

$$\Phi(t,\tau)=\begin{cases}\varphi\left(\dfrac{4t_1}{\tau+1},t_2,\dots,t_n\right), & 0\leqslant t_1\leqslant\dfrac{\tau+1}{4},\\ \psi(4t_1-\tau-1,t_2,\dots,t_n), & \dfrac{\tau+1}{4}\leqslant t_1\leqslant\dfrac{\tau+2}{4},\\ \mu\left(\dfrac{4t_1-2-\tau}{2-\tau},t_2,\dots,t_n\right), & \dfrac{\tau+2}{4}\leqslant t_1\leqslant 1.\end{cases}$$

From the diagrammatic point of view, this homotopy is explained quite simply (Fig. 59).

Now, to show that if $n>1$, then $[\varphi]+[\psi]=[\psi]+[\varphi]$, remember that

$$(\varphi+\psi)(t)=\begin{cases}\varphi(2t_1,t_2,\dots,t_n) & \text{when } 0\leqslant t_1\leqslant 1/2,\\ \psi(2t_1-1,t_2,\dots,t_n) & \text{when } 1/2\leqslant t_1\leqslant 1,\end{cases}$$

$$(\psi+\varphi)(t)=\begin{cases}\psi(2t_1,t_2,\dots,t_n) & \text{when } 0\leqslant t_1\leqslant 1/2,\\ \varphi(2t_1-1,t_2,\dots,t_n) & \text{when } 1/2\leqslant t_1\leqslant 1.\end{cases}$$

We shall verify that the mappings $\varphi+\psi$ and $\psi+\varphi$ are homotopic to the same mapping. (Hence, it follows that they are homotopic to one another.) Consider the homotopy $\Phi_1(t,\tau)$:

$$\Phi_1(t,\tau)=\begin{cases}\left.\begin{array}{ll} x_0, & 0\leqslant t_2\leqslant\tau/2\\ \varphi\left(2t_1,\dfrac{2t_2-\tau}{2-\tau},t_3,\dots,t_n\right), & \tau/2\leqslant t_2\leqslant 1\end{array}\right\} & 0\leqslant t_1\leqslant 1/2,\\ \left.\begin{array}{ll} \psi\left(2t_1-1,\dfrac{2t_2}{2-\tau},t_3,\dots,t_n\right), & 0\leqslant t_2\leqslant 1-\dfrac{\tau}{2}\\ x_0, & 1-\dfrac{\tau}{2}\leqslant t_2\leqslant 1\end{array}\right\} & 1/2\leqslant t_1\leqslant 1.\end{cases}$$

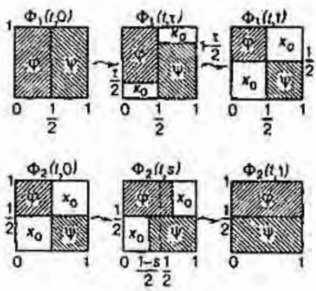

Fig. 60

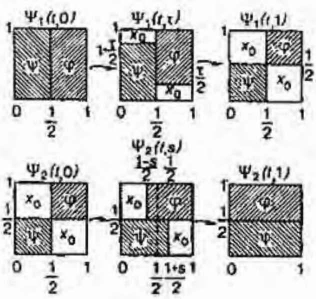

Fig. 61

It is easy to see that $\Phi_1(t, 0) = \varphi + \psi$, and

$$\Phi_1(t, 1) = \begin{cases} x_0, & 0 \leqslant t_2 \leqslant 1/2 \\ \varphi(2t_1, 2t_2 - 1, t_3, \ldots, t_n), & 1/2 \leqslant t_2 \leqslant 1 \end{cases} \quad 0 \leqslant t_1 \leqslant 1/2,$$
$$\qquad\qquad \begin{cases} \psi(2t_1 - 1, 2t_2, t_3, \ldots, t_n), & 0 \leqslant t_2 \leqslant 1/2 \\ x_0, & 1/2 \leqslant t_2 \leqslant 1 \end{cases} \quad 1/2 \leqslant t_1 \leqslant 1.$$

Consider another homotopy Φ_2

$$\Phi_2(t, s) = \begin{cases} \left.\begin{array}{ll} \varphi\left(\dfrac{2t_1}{1+s}, 2t_2 - 1, t_3, \ldots, t_n\right), & 0 \leqslant t_1 \leqslant \dfrac{1+s}{2} \\ x_0, & \dfrac{1+s}{2} \leqslant t_1 \leqslant 1 \end{array}\right\} \dfrac{1}{2} \leqslant t_2 \leqslant 1, \\ \left.\begin{array}{ll} x_0, & 0 \leqslant t_1 \leqslant \dfrac{1-s}{2} \\ \psi\left(\dfrac{2t_1 - 1 + s}{1+s}, 2t_2, t_3, \ldots, t_n\right), & \dfrac{1-s}{2} \leqslant t_1 \leqslant 1 \end{array}\right\} 0 \leqslant t_2 \leqslant \dfrac{1}{2}. \end{cases}$$

It is easy to verify that $\Phi_2(t, 0) = \Phi_1(t, 1)$, and

$$\Phi_2(t,1)=\begin{cases}\varphi(t_1,2t_2-1,t_3,\dots,t_n), & 0\leqslant t_1\leqslant 1,\ \frac{1}{2}\leqslant t_2\leqslant 1,\\ \psi(t_1,2t_2,t_3,\dots,t_n), & 0\leqslant t_1\leqslant 1,\ 0\leqslant t_2\leqslant \frac{1}{2}.\end{cases}$$

The homotopies Φ_1, Φ_2 are represented in Fig. 60 as diagrams.
Thus, we have

$$\varphi+\psi\sim\Phi_1(t,1)=\Phi_2(t,0)\sim\Phi_2(t,1). \qquad (*)$$

We perform a similar construction for the sum $\psi+\varphi$. Let us write out the homotopies:

$$\Psi_1(t,\tau)=\begin{cases}\left.\begin{array}{ll}\psi\left(2t_1,\dfrac{2t_2}{2-\tau},t_3,\dots,t_n\right), & 0\leqslant t_2\leqslant 1-\dfrac{\tau}{2}\\ x_0, & 1-\dfrac{\tau}{2}\leqslant t_2\leqslant 1\end{array}\right\}\ 0\leqslant t_1\leqslant\dfrac{1}{2},\\ \left.\begin{array}{ll}x_0, & 0\leqslant t_2\leqslant\dfrac{\tau}{2}\\ \varphi\left(2t_1-1,\dfrac{2t_2-\tau}{2-\tau},t_3,\dots,t_n\right), & \dfrac{\tau}{2}\leqslant t_2\leqslant 1\end{array}\right\}\ \dfrac{1}{2}\leqslant t_1\leqslant 1.\end{cases}$$

It is easy to see that $\Psi_1(t,0)=\psi+\varphi$, and

$$\Psi_1(t,1)=\begin{cases}\left.\begin{array}{ll}\psi(2t_1,2t_2,t_3,\dots,t_n), & 0\leqslant t_2\leqslant\dfrac{1}{2}\\ x_0, & \dfrac{1}{2}\leqslant t_2\leqslant 1\end{array}\right\}\ 0\leqslant t_1\leqslant\dfrac{1}{2},\\ \left.\begin{array}{ll}x_0, & 0\leqslant t_2\leqslant\dfrac{1}{2}\\ \varphi(2t_1-1,2t_2-1,t_3,\dots,t_n), & \dfrac{1}{2}\leqslant t_2\leqslant 1\end{array}\right\}\ \dfrac{1}{2}\leqslant t_1\leqslant 1.\end{cases}$$

It follows from another homotopy

$$\Psi_2(t,s) = \begin{cases} \varphi\left(\dfrac{2t_1 - 1 + s}{1 + s}, 2t_2 - 1, t_3, \dots, t_n\right), & \dfrac{1-s}{2} \leqslant t_1 \leqslant 1 \\ x_0, & 0 \leqslant t_1 \leqslant \dfrac{1-s}{2} \end{cases} \Bigg\} \frac{1}{2} \leqslant t_2 \leqslant 1,$$
$$\begin{cases} x_0, & \dfrac{1+s}{2} \leqslant t_1 \leqslant 1 \\ \psi\left(\dfrac{2t_1}{1+s}, 2t_2, t_3, \dots, t_n\right), & 0 \leqslant t_1 \leqslant \dfrac{1+s}{2} \end{cases} \Bigg\} 0 \leqslant t_2 \leqslant \frac{1}{2},$$

that $\Psi_2(t, 0) = \Psi_1(t, 1)$ and $\Psi_2(t, 1) = \Phi_2(t, 1)$.

The homotopies $\Psi_1(t, \tau)$, $\Psi_2(t, s)$ are represented in Fig. 61 as diagrams. We find that

$$\psi + \varphi \sim \Psi_1(t, 1) = \Psi_2(t, 0) \sim \Psi_2(t, 1) = \Phi_2(t, 1)$$

Finally, we obtain from the last chain of homotopies and the chain (*) that

$$\varphi + \psi \sim \Phi_2(t, 1), \psi + \varphi \sim \Phi_2(t, 1),$$

therefore $\varphi + \psi \sim \psi + \varphi$. ■

THEOREM 2. *Any mapping* $f : (X, x_0) \to (Y, y_0)$ *induces the group homomorphism* $\pi_{(I^n, \partial I^n)}(f) : \pi_n(X, x_0) \to \pi_n(Y, y_0)$.

THE PROOF is left to the reader.

Hint: Use the construction in Exercise 3°

The homomorphism $\pi_{(I^n, \partial I^n)}(f)$ is denoted by f_n and called the n-dimensional homotopy group ***homomorphism induced by the continuous mapping*** f.

Thus, the functor π_n, $n > 1$, acts from the category of base point spaces and their continuous mappings to the category of Abelian groups and their homomorphisms. Therefore, if

$$f : (X, x_0) \to (Y, y_0), g : (Y, y_0) \to (Z, z_0)$$

are continuous mappings then $(gf)_n = g_n f_n$, where f_n, g_n, $(gf)_n$ are the corresponding homomorphisms of n-dimensional homotopy groups.

2. **The Fundamental Group.** It will be interesting to consider separately the set

$$\pi_1(X, x_0) = \pi(I, \partial I; X, x_0) = \pi(S^1, p_0; X, x_0)$$

which is endowed with a group structure in the same manner as π_n, $n > 1$, and is applied in many problems. By general definition, each element of $\pi_1(X, x_0)$ is a homotopy class $[\varphi]$ of a certain mapping $\varphi : (I, \partial I) \to (X, x_0)$, where the image $\varphi(I)$ is a loop in the space X, starting and ending at the point x_0 (Fig. 62). The direction for circumnavigating the loop is given by a parameter $t \in I$. The product $\varphi \cdot \psi$ of two such loops φ and ψ is defined as a loop in X such that the image $(\varphi \cdot \psi)(t)$

Fig. 62

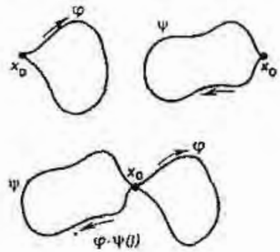

Fig. 63

runs over the loop φ as the parameter t changes from 0 to 1/2, and the image $(\varphi \cdot \psi)(t)$ runs over the loop ψ (Fig. 63) as t ranges from 1/2 to 1, viz.,

$$(\psi \cdot \varphi)(t) = \begin{cases} \varphi(2t), & 0 \leqslant t \leqslant 1/2; \\ \psi(2t - 1), & 1/2 \leqslant t \leqslant 1. \end{cases}$$

As can be seen, the product of loops is defined in much the same manner as the sum of spheroids. The difference in terms (i.e., sum and product) is explained by the generally accepted custom of employing additive notation (i.e., the '+' sign) for Abelian groups. The composition of loops described above is not always commutative. Therefore, the product $[\varphi] \cdot [\psi] = [\varphi \cdot \psi]$ (generally speaking, not commutative) may be defined on the group $\pi_1(X, x_0)$.

Exercise 9°. Verify that the group π_1 of the wedge of two circumferences is not commutative.

DEFINITION 2. The group $\pi_1(X, x_0)$ is called the *fundamental group* of a topological space X with a base point x_0.

PROPOSITION. *The set $\pi_1(X, x_0)$ is a group under the described product operation of the product.*

PROOF. Note that in the proof of Theorem 1, the condition $n > 1$ was used only while proving the commutativity of the group π_n, where the second coordinate of the spheroid was taking part in the necessary homotopies. Therefore all the previous steps of the proof for Theorem 1 can be used for $\pi_1(X, x_0)$ without introducing any changes. In doing so, the unit and inverse elements in $\pi_1(X, x_0)$ are defined exactly in the same way, viz., $\theta = [\varphi_0]$, where $\varphi_0(I) = x_0$ is a constant loop; for each $[\varphi] \in \pi_1(X, x_0)$, $[\varphi]^{-1} = [\varphi^{-1}]$, where $\varphi^{-1}(t) = \varphi(1 - t)$ is the loop circumnavigated in the reverse direction. Thus, the required statement follows directly from the proof of Theorem 1. ■

In order to distinguish the difference between the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ of the same space having different base points $x_0 \in X$, $x_1 \in X$, we shall need some more concepts.

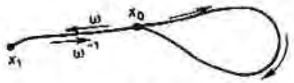

Fig. 64

The *product* $\omega_1 \cdot \omega_2$ of paths* ω_1 and ω_2 such that $\omega_2(0) = \omega_1(1)$ is defined in the same way as the product of loops:

$$(\omega_1 \cdot \omega_2)(t) = \begin{cases} \omega_1(2t), & 0 \leqslant t \leqslant 1/2, \\ \omega_2(2t - 1), & 1/2 \leqslant t \leqslant 1. \end{cases}$$

It is evident that $\omega_1 \cdot \omega_2$ is a path in the space X. A *constant path* in X is a path $C_{x_0}: I \to X$ such that $C_{x_0}(t) \equiv x_0$ for $t \in [0, 1]$. The *reverse path* of a path ω is a path $\omega^{-1}: I \to X$ such that $\omega^{-1}(t) = \omega(1 - t)$. Since $(\omega^{-1} \cdot \omega)(0) = (\omega^{-1} \cdot \omega)(1)$, the path $(\omega^{-1} \cdot \omega)(t)$ is a loop at the point $\omega(0)$.

Exercise 10°. Draw the path $(\omega^{-1} \cdot \omega)(t)$. Show that $[\omega^{-1} \cdot \omega] = \Theta$ in $\pi_1(X, x_0)$.

THEOREM 3. *Any path $\omega : I \to X$ joining points x_0 and x_1, i.e., $\omega(0) = x_0$, $\omega(1) = x_1$, induces the isomorphism of groups*

$$S_1^\omega : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

which depends only on the homotopy class of the path ω.

PROOF. Let $[\varphi] \in \pi_1(X, x_0)$. Consider the mapping $\psi : (I, \partial I) \to (X, x_1)$ which is given by the formula

$$\psi(t) = \begin{cases} \omega(1 - 3t) \text{ if } 0 \leqslant t \leqslant 1/3, \\ \varphi(3t - 1) \text{ if } 1/3 \leqslant t \leqslant 2/3, \\ \omega(3t - 2) \text{ if } 2/3 \leqslant t \leqslant 1. \end{cases}$$

The path $\psi(t)$ can be represented visually as the loop $\psi(t) = (\omega^{-1} \cdot \varphi \cdot \omega)(t)$ (Fig. 64); we thus associate each element $[\varphi] \in \pi_1(X, x_0)$ with an element $[\psi] \in \pi_1(X, x_1)$, and obtain a mapping $S_1^\omega : \pi_1(X, x_0) \to \pi_1(X, x_1)$. S_1^ω happens to be a group homomorphism. (Verify!)

Similarly, we associate each element $[\psi] \in \pi(X, x_1)$ with an element $[\varphi] \in \pi(X, x_0)$, where

$$\varphi(t) = (\omega \cdot \psi \cdot \omega^{-1})(t) = \begin{cases} \omega(3t) & \text{if } 0 \leqslant t \leqslant 1/3, \\ \psi(3t - 1) & \text{if } 1/3 \leqslant t \leqslant 2/3, \\ \omega(3 - 3t) & \text{if } 2/3 \leqslant t \leqslant 1. \end{cases}$$

We thus obtain a mapping

$$S_1^{\omega^{-1}} : \pi_1(X, x_1) \in \pi_1(X, x_0).$$

Exercise 11°. Show that $S_1^{\omega^{-1}}$ is a group homomorphism and that the homomorphisms $S_1^{\omega^{-1}}$ and S_1^ω are reciprocal, i.e., $S_1^\varphi = (S_1^{\omega^{-1}})^{-1}$.

---

* Remember that a path in a space X is a continuous mapping of a line-segment, $\omega: I \to X$.

Thus, S_1^ω is an isomorphism. It is clear from its construction that it remains unaltered under a fixed-end homotopy of the path ω. ∎

Exercise 12°. Prove that if $f : X \to Y$ is a continuous mapping, then for any path ω joining points x_0 and x_1, the diagram

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_1"] \arrow[d, "S_1^\omega"'] & \pi_1(Y, f(x_0)) \arrow[d, "S_1^{\bar\omega}"] \\
\pi_1(X, x_1) \arrow[r, "f_1"'] & \pi_1(Y, f(x_1))
\end{tikzcd}$$

is commutative. Here $\bar\omega = f\omega$ is a path joining the points $f(x_0)$ and $f(x_1)$.

It follows at once from Theorem 3 that if a space X is path-connected, then the groups $\pi_1(X, x_0)$ at different points $x_0 \in X$ are isomorphic to each other and can be considered as one abstract group $\pi_1(X)$. This group is called the *fundamental group* of the path-connected space X.

We shall now adduce another fact that follows from Theorem 3.

COROLLARY. *Any element* $[\alpha] \in \pi_1(X, x_0)$ *defines an automorphism* $S_1^{[\alpha]}$ *of the group* $\pi_1(X, x_0)$.

PROOF. In virtue of Theorem 3, there is an isomorphism $S_1^{[\alpha]} : \pi_1(X, x_0) \to \pi_1(X, x_0)$, since α is a loop at the point x_0. In addition, the isomorphism S_1^α only depends on the homotopy class of the path α. ∎

An important class of spaces is singled out by the following definition.

DEFINITION 3. A path-connected space X is said to be *1-connected* if any two paths $\omega_1 : I \to X$ and $\omega_2 : I \to X$ such that $\omega_1(0) = \omega_2(0) = x_0$, $\omega_1(1) = \omega_2(1) = x_1$ belong to the same homotopy class in $\pi_1(I, \partial I: X, x_0 \cup x_1)$, i.e., are homotopic in the class of paths starting at x_0 and ending at x_1.

THEOREM 4. *A path-connected space X is 1-connected if and only if* $\pi_1(X) = 0$.

This theorem describes 1-connected spaces in terms of their fundamental groups. The proof is easy, and we shall skip it.

Exercises.

13°. Verify that the Euclidean space R^n is 1-connected, and S^1 and the torus $S^1 \times S^1$ are not 1-connected.

14°. Construct an example of a connected space with non-isomorphic groups $\pi_1(X, x_0)$ at different points x_0.

Hint: Use the example of a connected, but not path-connected space from Sec. 10, Ch. II.

We shall now investigate how higher homotopy groups depend on a variation of a base point. The homotopy group $\pi_n(X, x_0)$ turns out to vary in the same way as the fundamental group $\pi_1(X, x_0)$ when its base point changes.

THEOREM 5. *Any path $\omega : I \to X$ joining points x_0 and x_1 determines the isomorphism*

$$S_n^\omega : \pi_n(X, x_1) \to \pi_n(X, x_0)$$

depending on the homotopy class $[\omega] \in \pi(I, \partial I; X, x_0 \cup x_1)$. *In addition, for any mapping* $f : X \to Y$, *the diagram*

$$\begin{tikzcd} \pi_n(X,x_1) \arrow[r, "f_n"] \arrow[d, "S_n^{\omega}"'] & \pi_n(Y,y_1) \arrow[d, "S_n^{\bar\omega}"] \\ \pi_n(X,x_0) \arrow[r, "f_n"'] & \pi_n(Y,y_0) \end{tikzcd},$$

in which $S_n^{\bar\omega}$ *is an isomorphism determined by the path* $\bar\omega = f\omega$ *between the points* $y_0 = f(x_0)$, $y_1 = f(x_1)$ *is commutative.*

We will just outline the idea behind the proof of this theorem. Let $[\varphi] \in \pi_n(X, x_1)$. As was in the case of the fundamental group, the element $[\varphi]$ is associated with an element $[\psi] \in \pi_n(X, x_0)$. This procedure can be represented visually as pulling a 'whisker' out of the spheroid at the point x_1 to the point $\omega(t)$ and extending it along the path ω to the point x_0 (Fig. 65).

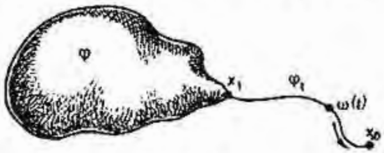

Fig. 65

Thus, we obtain a mapping $S_n^{\omega} : \pi_n(X, x_1) \to \pi_n(X, x_0)$ which is an isomorphism with the required properties. Here, we have omitted all the details.

A corollary to Theorem 5 states that any element $[\alpha] \in \pi_1(X, x_0)$ determines an automorphism of the group $\pi_n(X, x_0)$.

Thus, the group $\pi_1(X, x_0)$ acts on the group $\pi_n(X, x_0)$ as the group of automorphisms.

It is now natural to define the following generalization of 1-connected spaces.

DEFINITION 4. If, for a space X and any points $x_0, x_1 \in X$, lying in the same path component, the isomorphism $S_n^{\omega}: \pi_n(X, x_1) \to \pi_n(X, x_0)$ does not depend on the choice of the path ω joining x_0 to x_1 then the space X is said to be *n-simple* (or *homotopy simple in dimension n*).

Exercise 15°. Verify that a 1-connected space is 1-simple.

We leave the proof of the following statement to the reader.

THEOREM 6. *A space* X *is n-simple if and only if for any point* $x_0 \in X$, *the group* $\pi_1(X, x_0)$ *acts trivially on* $\pi_n(X, x_0)$, i.e., *does not alter the elements of* $\pi_n(X, x_0)$.

It immediately follows from Theorem 4 that a 1-connected space is n-simple for all $n \geqslant 1$.

4. COMPUTING THE FUNDAMENTAL AND HOMOTOPY GROUPS OF SOME SPACES

In this section, the fundamental group of the circumference, and also of an arbitrary M_p- or N_q-type closed surface will be calculated. The necessary combinatorial technique is based on the results of Sec. 4, Ch. II, and given at the begin-

ning of the section (see Items 1 and 2). Meanwhile, the topological invariance of the Euler characteristic of a closed surface (see Item 5) is established. Further, the problem of computing higher homotopy groups is discussed, and their application to a problem concerning the fixed points of a continuous mapping is given (the Brouwer theorem and the fundamental theorem of algebra).

1. Line Paths on a Surface and Their Combinatorial Homotopies. Consider a closed surface X given, as in Sec. 4, Ch. II, by its subdivision. This means that a development Π is given, and the surface X is homeomorphic to the factor space Π/R, where R is an equivalence determined by the gluing homeomorphisms of the development.

Denote the product of the residue class mapping $\alpha\colon \Pi \to \Pi/R$ and the homeomorphism $\beta\colon \Pi/R \to X$ by $\varkappa$. Then the mapping $\varkappa\colon \Pi \to X$ is the one determining the subdivision of X into the images of polygons, edges and vertices of the development (we will call the $\varkappa$-images of edges the *edges*, and the $\varkappa$-images of vertices the vertices of the subdivision). An edge of a subdivision is the $\varkappa$-image of two edges, a and a^{-1}, or a and a. We will denote it by the letter a; the $\varkappa$-image of a vertex A will be denoted by the same letter A; and we will call *interior points* of an edge its points which are different from the vertices.

We shall require the following two elementary operations over subdivisions: (a) adding a new vertex: an interior point of an edge is declared to be a new vertex of the subdivision; (b) adding a new edge: one of the polygons of the development is subdivided into two by its diagonal. The $\varkappa$-image of this diagonal in X is declared to be a new edge of the development.

Consider an edge a in the development Π, and let $\gamma\colon I \to a$ be an affine mapping (linear path) under which the points 0 and 1 are mapped into the vertices of the edge. Then the mapping $\tilde{\gamma} = \varkappa\gamma : I \to X$ determines a path on the surface X, which we will call the *elementary path*. It is evident that the image of an elementary path either coincides with one of the vertices of the edge a of the subdivision of the surface or completely covers the edge. In the first case, an elementary path is constant and considered to be zero ($\tilde{\gamma} = 0$). In the second case, the beginning of the linear path γ either coincides with the beginning of the oriented edge a, or with its end. Accordingly, we will denote an elementary path by a or a^{-1} ($\tilde{\gamma} = a$ or $\tilde{\gamma} = a^{-1}$, respectively). We will use the same notation for $\tilde{\gamma}$ if $\gamma : I \to a^{-1}$, assuming that $(a^{-1})^{-1} = a$.

Thus, to each oriented edge $a(a^{-1})$ of the development, there corresponds an elementary path $a(a^{-1})$ in the subdivision.

DEFINITION 1. A finite product of elementary paths in a subdivision Π of a surface X is called a *line path*. A closed line path is called a *line loop*.

By Definition 1, a line path λ can be written in the form of the product of elementary paths $\lambda = \lambda_1\lambda_2 \ldots \lambda_s$, where $\lambda_i = a_i^{\pm 1}$ or $\lambda_i = 0$. Omitting zeroes, we associate the path λ with the word $\omega(\lambda) = a_{i_1}^{\pm 1} \ldots a_{i_s}^{\pm 1}$ indicating the order and direction of circumnavigation of the edges of the surface X along the path λ.

Consider the boundary Γ_i of a polygon Q_i of a development Π. By associating each edge of the boundary with an elementary path as described above, we shall

associate the whole boundary with the line path λ_i in X, determined by the word $\omega(\lambda_i) = \omega(Q_i)$, whereas the word $\omega(Q_i)$ in turn describes a plan for gluing the polygon Q_i (see Item 2, Sec. 4, Ch. II).

For example, a line path λ corresponding to the oriented boundary of the polygon Q which represents the torus development (see Fig. 41) is determined by the word $\omega(\lambda) = aba^{-1}b^{-1}$.

DEFINITION 2. A *combinatorial deformation* of Type I (Type II, respectively) of a line loop λ is an introduction or deletion of an aa^{-1}-type combination (or the word $\omega(Q_i)$) into the word $\omega(\lambda)$, respectively, determining that line loop in X which corresponds to the oriented boundary of the polygon Q_i of the development Π.

DEFINITION 3. Line paths γ and γ' in Π are said to be *combinatorially homotopic* in Π if one is obtained from the other using a finite number of Type I or II combinatorial deformations.

Note that any line path in a subdivision Π of a surface X can be considered as a line path in a subdivision Π_1 which is obtained from Π by applying a finite number of (a)- or (b)-type operations.

LEMMA 1. *Let a subdivision Π_1 be obtained from a subdivision Π by applying a finite number of (a)- or (b)-type operations. Then for any line loop λ in Π_1, there exists a line loop λ' in Π, which is combinatorially homotopic in Π_1 to the loop λ.*

PROOF. It is obvious that it suffices to consider the case when Π_1 is derived from Π by having applied one of the operations (a) or (b). Let Π_1 be obtained from Π by subdividing an edge a into two new edges b and c (the operation of adding a new vertex having been applied). If the loop λ contains one of the combinations bb^{-1}, cc^{-1}, $b^{-1}b$, $c^{-1}c$, then it can be omitted having obtained a loop which is homotopic to λ. Having omitted all such combinations, we obtain a loop either not containing $b^{\pm 1}$, $c^{\pm 1}$ at all or containing them in the form $bc(= a)$ or $c^{-1}b^{-1}(= a^{-1})$; in either case, it is the required line path λ' from Π.

Now, let Π_1 be obtained from Π by adding a new edge d which subdivides a certain polygon in Π_1 into parts E and F. Let the boundary paths of E and F be ud^{-1} and dv, respectively (Fig. 66). If the line loop λ includes the edge $d^{\pm 1}$, then we replace it by the path $v^{\mp 1}$ (or $u^{\pm 1}$). The loop λ' thus obtained is combinatorially homotopic to λ and is a line loop from Π. ■

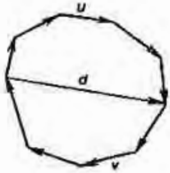

Fig. 66

LEMMA 2. *Let Π_1 be obtained from Π by an (a)- or (b)-type operation. Then any line loop λ in Π which is combinatorially homotopic to zero in Π_1, will also be combinatorially homotopic to zero in Π.*

PROOF. By the data given, there exists a sequence of line loops $\lambda = v_0, v_1, \dots, v_r = 0$ in Π_1, where v_{i+1} is obtained from v_i by means of one combinatorial deformation. In addition, $v_1, \dots, v_r$ are not, generally speaking, loops in Π. For each loop v_i, $i = 1, \dots, r$, we construct a line loop ω_i homotopic to it in Π so that each loop ω_{i+1} in the sequence of loops $\lambda, \omega_1, \dots, \omega_r = 0$, is obtained from ω_i via one or more combinatorial deformations.

Assume that Π_1 is obtained from Π by subdividing an edge a into edges b and c (i.e., by an (a)-type operation). Then we associate each loop v_i with the loop ω_i, having assigned to an edge different from $b^{\pm 1}$ and $c^{\pm 1}$ the same edge, the edge $a^{\pm 1}$ to the edge $b^{\pm 1}$, and nothing to the edge $c^{\pm 1}$. It is easy to verify that then the transfer from ω_i to ω_{i+1}, $i = 1, \dots, r$ is performed by a Type I or II combinatorial deformation.

If, however, Π_1 is obtained from Π by a (b)-type operation, then we associate any edge different from the subdividing edge d with itself, and replace $d(d^{-1})$ by the path $u(u^{-1})$. If, now, to obtain v_{i+1}, we insert or delete the combination dd^{-1} in v_i, then the combination uu^{-1} should be inserted or deleted, respectively, in ω_i. Type II deformations in Π_1 will correspond to Type I or Type II deformations in Π. ■

2. Combinatorial Approximations of Paths and Homotopies.

We will show here that any continuous path in a triangulation K is homotopic to a line path, and also study the relationship between combinatorial and continuous homotopies.

Hereafter, we consider only fixed-end homotopies of paths and loops.

LEMMA 3. *Let a triangulation K of a surface X be given. Let $\lambda: I \in K$ be a continuous path in K, $\lambda(0)$, $\lambda(1)$ being the vertices of the triangulation. Then there exists a line path in K, which is homotopic to it.*

PROOF. Subdivide the line-segment $I = [0, 1]$ with a finite number of points $\{t_k\}_{k=0}^{n}$ ($t_0 = 0, t_n = 1$) into sufficiently small line-segments so that for each interval (t_{k-1}, t_{k+1}), $k = 1, \dots, n-1$, there may be a vertex $A_k \to K$ such that the image $\lambda(t_{k-1}, t_{k+1})$ of the interval may lie wholly in the star $S(A_k)$, the union of the open triangles and edges of the triangulation K adjacent to a certain vertex A_k and the vertex A_k itself. Since $S(A_k)$ is an open set in X, and λ is a continuous mapping, this can always be achieved (see Ex. 7, Sec. 13, Ch. II).

Now, we associate each point $t_k \in I$ with the vertex $A_k \in K$. Note, moreover, that for any $k = 1, \dots, n-1$,

$$\lambda((t_k, t_{k+1})) \subset S(A_k) \cap S(A_{k+1}),$$

where $S(A_k) \cap S(A_{k+1})$ obviously contains the triangle which is adjacent to both A_k and A_{k+1}. Therefore, if $A_k \neq A_{k+1}$ then they are joined in K by an edge which we will denote by l_k. Let $\lambda_k': [t_k, t_{k+1}] \to l_k$ be an elementary path which is the extension of the indicated correspondence of the vertices and points t_k, t_{k+1}. If $A_k = A_{k+1}$ then we consider λ_k' to be equal to zero. The product of elementary

paths λ_k' determines a line path $\lambda' : I \to K$ called a *line approximation of the path*.

The paths λ and λ' are homotopic to one another. In fact, in virtue of the structure of the path λ', for any point $t \in I$, the images $\lambda(t)$ and $\lambda'(t)$ lie in the same closed topological triangle from K. Therefore, they can be joined by a 'line-segment', the homeomorphic image of a line-segment in a triangle of the development; consequently, it is natural to give a linear deformation of the point $\lambda(t)$ into the point $\lambda'(t)$ which determines the required homotopy. Note, moreover, that point $\lambda(t)$ does not leave that closed triangle, edge or vertex, in which it initially was in the course of the homotopy. ■

It is necessary to distinguish between line loops which are homotopic to a constant one in the topological or combinatorial sense. We will call a loop which is homotopic to a constant one *contractible* or *combinatorially contractible loop*, respectively.

LEMMA 4. *A contractible line loop λ in a triangulation K is combinatorially contractible in K.*

PROOF. Let a line loop λ be given by a mapping of a line-segment $\psi : I_1 \to K$. Let $F : I_1 \times I_2 \to K$ be the contraction of the loop to a vertex $x_0 \in K$, i.e.,

$$F|_{I_1 \times \{0\}} = \psi,\ F|_{I_1 \times \{1\}} = c_0 : I_1 \to x_0 \in K.$$

It is clear that $F|_{\{0\} \times I_2} : I_2 \to x_0$ and $F|_{\{1\} \times I_2} : I_2 \to x_0$.

Since F is a contraction keeping the ends of the loop fixed, the edges AB, CD and BD (Fig. 67) are mapped into one point x_0. We mark those points on AB whose images are the vertices of K, and draw vertical straight lines through them. Then, by drawing additionally other vertical and horizontal lines and diagonals (Fig. 67), we will obtain a sufficiently fine triangulation Σ of the square $ABCD$ for the image of the star $S(V)$ of the triangulation Σ under the mapping F to lie in the star $S(W)$ of a certain vertex of the triangulation K (this follows from Ex. 7, Sec. 13, Ch. II).

We now associate the vertex V with the vertex W and perform a similar operation over all the vertices of the triangulation Σ. Then we extend this mapping to the edges of the triangulation Σ in precisely the same manner as we did in the proof for the lemma on a line approximation of a path. The mapping which we obtain, i.e., $F_1 : \Sigma_1 \to K$, where Σ_1 is the union of the edges of the triangulation Σ, transforms the subdivided side AB into a certain line loop $\bar{\lambda}$ in K.

We now show that $\bar{\lambda}$ is combinatorially deformable into λ. In fact, during a line approximation, no point of a path leaves the triangle, edge or vertex in which it was

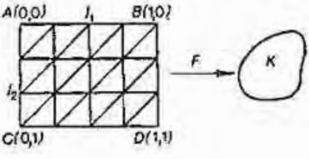

Fig. 67

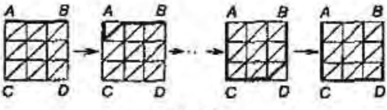

Fig. 68

positioned. Therefore, the loop $\bar{\lambda}$ consists of the same elementary paths as λ (if the null paths are neglected). However, generally speaking, some edges can be run several times in different directions. Thus, we can make a transfer from $\bar{\lambda}$ to λ by Type I combinatorial deformations.

Note now that in the triangulation Σ, the subdivided side AB can be transformed into the subdivided broken line $ACDB$ via combinatorial deformations of Type I and II by successive 'squeezings' of a single triangle (Fig. 68). However, each of these combinatorial deformations applied to AB determines, due to the structure of the mapping F_1, its own Type I or II combinatorial deformation of the loop $\bar{\lambda}$ in K (verify!).

Thus, we have shown that by means of Type I and II combinatorial deformations, the line loop λ can be transformed into the loop $\bar{\lambda}$, and then into the F_1-image of the path $ACDB$. But this image is the point x_0 and therefore λ is combinatorially homotopic to a constant. ■

We leave the proof of another two uncomplicated statements which we will use later to the reader.

Exercises.

1°. Prove that a line path λ in a subdivision Π, determined by the word $\omega(\lambda) = aa^{-1}$, is homotopic to a constant path.

2°. Prove that a line path in a subdivision Π equal to the image of the boundary of some polygon of the development Π is homotopic in X to a constant path.

It follows from Exercises 1 and 2 that any combinatorial homotopy determines a usual continuous homotopy between line paths.

NOTE. In the next item, we shall have recourse to a special case of the combinatorial technique which we developed above, viz., subdividing the circumference S^1.

We fix a finite number of points $A', B', C', \ldots$ on S^1 and specify a homeomorphism φ of the boundary of a convex polygon $ABC \ldots$ in S^1 so that $\varphi(A) = A'$, $\varphi(B) = B'$, $\varphi(C) = C'$ ... . We will say that the homeomorphism φ determines a subdivision of S^1 with the edges $\overset{\frown}{A'B'} = \varphi(\overline{AB})$, $\overset{\frown}{B'C'} = \varphi(\overline{BC})$, $\overset{\frown}{C'A'} = \varphi(\overline{CA})$, ... and vertices $A', B', C', \ldots$. Line paths and Type I combinatorial deformations are defined naturally here. It is easy to see that Lemmata 1-4 remain valid for such subdivisions, with the operations over (b)-type subdivisions and Type II combinatorial deformations vanishing.

3. The Fundamental Group of a Circumference.

We now can calculate the group $\pi_1(S^1)$.

THEOREM 1. *The group $\pi_1(S^1)$ is Abelian and isomorphic to the group Z.*

To prove this theorem, we shall require the following auxiliary statement which will be strengthened later (see Theorem 4, Sec. 4, of the present chapter).

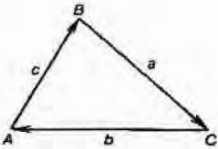

Fig. 69

LEMMA 5. *The fundamental groups of homeomorphic spaces are isomorphic.*

PROOF. Let X, Y be topological spaces with base points x_0, y_0, respectively, and $\varphi : (X, x_0) \to (Y, y_0)$ a homeomorphism. Then the homeomorphisms of the fundamental groups are defined:

$$\pi(\varphi) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

and

$$\pi(\varphi^{-1}) : \pi_1(Y, y_0) \to \pi_1(X, x_0),$$

and due to the functorial property, we have

$$\pi(\varphi^{-1})\pi(\varphi) = \pi(\varphi^{-1}\varphi) = 1_{\pi_1(X, x_0)},$$
$$\pi(\varphi)\pi(\varphi^{-1}) = \pi(\varphi\varphi^{-1}) = 1_{\pi_1(Y, y_0)},$$

therefore, $\pi(\varphi) = [\pi(\varphi^{-1})]^{-1}$. ∎

THE PROOF OF THEOREM 1. By the last lemma, it suffices to compute the fundamental group of a plane triangle. Let Δ be a triangle with vertices A, B, C, oriented edges a, b, c and the base vertex A (Fig. 69).

We first compute the group $\pi_1(\Delta, A)$. Let λ be an arbitrary loop in Δ with the origin at the point A. According to Lemma 3, there exists a line loop λ' in the homotopy class of the loop λ. (It is clear that the triangle Δ is a subdivision.) Associating each edge a, b, c with loops $\bar{a}$, $\bar{b}$, $\bar{c}$ according to the following rule: $\bar{a} = cab$, $b = b^{-1}b$, $\bar{c} = cc^{-1}$, we show that the classes of the loops $\bar{a}$, $\bar{b}$, $\bar{c}$, which need not necessarily be different, are generators of the group $\pi_1(\Delta, A)$. Any line loop λ' consists of elementary paths that correspond to the edges, i.e., $\lambda' = \varphi(a, b, c)$. By replacing each edge by its corresponding loop in this expression, we obtain a new loop $\lambda'' = \varphi(\bar{a}, \bar{b}, \bar{c})$. It is easy to see that the loops λ' and λ'' are combinatorially homotopic.

In fact, this replacement of an edge by a loop makes us first 'reach' the origin of the edge from the fixed vertex A, and then, having passed through this edge, 'return' to A along the shortest path (Fig. 69). Therefore, during each successive replacement of an edge by a loop, we must, after returning to A from the end P of the previous edge, 'start' for the origin of the next edge, i.e., for the same point P. Thus, after this replacement, a $\Pi\Pi^{-1}$-form path is inserted between each two adjacent edges of the loop, i.e., a path which is combinatorially homotopic to zero.

Consequently, in the homotopy class of the loop λ', a line loop λ which is a finite product of the loops $\bar{a}$, $\bar{b}$, $\bar{c}$, and their inverses can always be found.

Note now that the loops $\bar{b}$, $\bar{c}$, are homotopic to constant loops. Therefore, the loop $\bar{a}$ (or, more precisely, the homotopy class determined by it in $\pi_1(\Delta, A)$) is a unique generator in the group $\pi_1(\Delta, A)$. The element $\bar{a}$ is non-trivial, because if the loop $\bar{a}$ were contractible then, by Lemma 4, it would also be combinatorially contractible, i.e.,reducible to zero by a finite number of combinatorial Type I deformations, which is obviously impossible. Consequently, the loop $\bar{a}$ is not combinatorially contractible and therefore determines a non-trivial element $\bar{a} \in \pi_1(\Delta, A)$. Similarly, any element $[\bar{a}^l] \in \pi_1(\Delta, A)$, where $l > 1$, is non-trivial.

Thus, $\pi_1(\Delta, A)$ is a free cyclic group generated by the element $[\bar{a}]$, i.e., an Abelian group isomorphic to Z. ■

Exercise 3°. By generalizing the structure of the proof of Theorem 1, prove that the fundamental group of the wedge of m circumferences is a free group with m generators.

The following theorem is a useful instrument for calculating the fundamental groups of more complicated spaces.

THEOREM 2 (VAN KAMPEN). *Let X be a topological space obtained as the union $X = X_1 \cup X_2$ of open subsets X_1 and X_2 such that the spaces X_1, X_2 and $X_0 = X_1 \cap X_2$ are path-connected and nonempty, and let $p \in X_0$. Consider the commutative diagram generated by the embedding mappings:*

$$\begin{tikzcd} & \pi_1(X_0,p) \arrow[dl, "\theta_1"'] \arrow[dr, "\theta_2"] \arrow[dd, "\omega_0"] & \\ \pi_1(X_1,p) \arrow[dr, "\omega_1"'] & & \pi_1(X_2,p) \arrow[dl, "\omega_2"] \\ & \pi_1(X,p) & \end{tikzcd}$$

Then the group $\pi_1(X, p)$ is a factor group of the free product $\pi_1(X_1, p) * \pi_1(X_2, p)$ by the normal subgroup generated by the set $\{\theta_1\alpha * \theta_2\alpha^{-1} : \alpha \in \pi_1(X_0, p)\}$. In other words, the group $\pi_1(X, p)$ is generated by the images of the elements $\pi_1(X_i, p)$, $i = 1, 2$, and the only relations between the generators are derived relations in each of the groups $\pi_1(X_i, p)$, $i = 1, 2$, and the relations $\omega_1\theta_1\alpha = \omega_2\theta_2\alpha$, where $\alpha \in \pi_1(X_0, p)$.

Exercises.

4°. Using the van Kampen theorem, derive the statement of Exercise 3.

5°. Calculate the fundamental group of the space consisting of two circumferences joined by line-segments (Fig. 70).

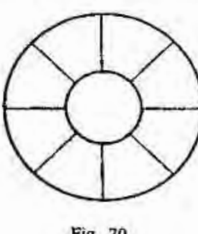

Fig. 70

4. The Fundamental Group of a Surface. Turning our attention to the fundamental groups of surfaces, we may assume on the basis of Lemma 5 that a closed surface can be given in subdivided form determined by a canonical development.

THEOREM 3. *Let X be a Type I or II closed surface determined by a word ω of the form $a_1b_1a_1^{-1}b_1^{-1} \dots a_pb_pa_p^{-1}b_p^{-1}$ or $a_1a_1a_2a_2 \dots a_qa_q$, respectively, and let $x_0 \in X$ be a certain point on the surface (i.e., triangulation vertex). Then $\pi_1(X, x_0)$ is a group with generators $a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_p$ or $a_1, a_2, \dots, a_q$, respectively, and one defining relation $\omega = e$, where e is the identity element.*

PROOF. Let X_1 be a closed surface, $\mathscr{P}$ its canonical development determined by a polygon Q and the word $\omega(Q)$. Let $X_1 = \varkappa(Q_1)$, where Q_1 is the union of all edges of the polygon Q. Since all the vertices of Q in the development $\mathscr{P}$ are equivalent, their images under the mapping $\varkappa$ coincide in X. Consequently, the image of each edge is homeomorphic to a circumference, and X_1 is the wedge of circumferences glued at the point x_0 which is the image of the vertices of the polygon Q. In addition, the number of the circumferences in the wedge equals $2p$ if the surface X has the type M_p, and q if X has the type N_q. It follows from one property of the fundamental group of the wedge of circumferences (see Ex. 3) that $\pi_1(X_1, x_0)$ is a free group generated by $a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_p$ if X has the type M_p or by $a_1, a_2, \dots, a_q$ if X has the type N_q. We denote this group by G.

We shall now consider the embedding mapping $i : X_1 \to X$ and the homomorphism of the fundamental groups, which is induced by it, viz.,

$$i_* : \pi_1(X_1, x_0) \to \pi_1(X, x_0).$$

We will calculate the group $\pi_1(X, x_0)$ as follows. First, we prove that i_* is an epimorphism. Then, using the theorem concerning epimorphisms, we obtain

$$\pi_1(X, x) \cong \pi_1(X, x)/\mathrm{Ker}\, i_* = G/\mathrm{Ker}\, i_*.$$

The calculation of the kernel $\mathrm{Ker}\, i_*$ will complete the proof of the theorem.

We first prove that i_* is an epimorphism. Let $\alpha \in \pi_1(X, x_0)$ and K some triangulation of the surface X. Then, by Lemma 3 concerning line approximations, there is a line loop λ (in the subdivision K) in the homotopy class of α. K may be assumed to be obtained from the canonical subdivision $\mathscr{P}$ of the surface X with the aid of a finite number of (a)- or (b)-type operations.

Therefore, due to Lemmata 1, 2 and Exercises 1, 2, in the same class of $\alpha \in \pi_1(X, x_0)$, there is a line loop λ' (i.e., made up of the edges of X_1) in the subdivision $\mathscr{P}$. Thus, a certain class $\beta_\alpha \in \pi_1(X, x_0)$ is defined for which it is obvious that $i_*(\beta_\alpha) = \alpha$. The surjectivity of i_* is therefore proved.

We now take up the task of calculating the kernel $\mathrm{Ker}\, i_*$ of the epimorphism i_*. Let $\gamma \in \mathrm{Ker}\, i_*$, and λ a line loop of the subdivision $\mathscr{P}$ from the class of γ. Then λ is obviously contractible to a point in X. According to Lemma 4, there exists in $\mathscr{P}$ a combinatorial contraction of λ to the vertex x_0. In other words, the word $\omega(\lambda)$ which determines the loop λ is reduced to the zero word by a finite number of Type I or II combinatorial deformations. It then becomes clear that the word $\omega(\lambda)$

may only consist of combinations of the form

(1) aa^{-1};

(2) or more complex, such as

$$\omega_4 h_2 \omega_1 h \omega^l h^{-1} \omega_2 h_1 \omega^m h_1^{-1} \omega_3 h_2^{-1} \omega_5, \tag{$*$}$$

where $\omega_4 \cdot \omega_5 = \omega$, $\omega_1 \cdot \omega_2 \cdot \omega_3 = \omega$; h_1, h_2, h are the words of the given subdivision, and l, m are integral exponents (they may be both positive and negative); and

(3) combinations similar to $(*)$, but with other partitions of the word ω into components. This follows from the given development not having any bounding words other than ω.

It is easy to see that Type I combinatorial deformations (i.e., additions or deletions of an aa^{-1}-type combination) do not take the loop λ out of its homotopy class, since the loop aa^{-1} is homotopic to zero in X_1. We may assume, thanks to this fact that there are no Type I combinations in $\omega(\lambda)$. The combinations of type $(*)$ are simplified by combinatorial Type I deformations as follows:

$$\begin{aligned}
(*) \Rightarrow\ & \omega_4 h_2 \omega_1 h \omega^l h^{-1} \omega_1^{-1} \omega_1 \omega_2 \omega_3 \omega_3^{-1} h_1 \omega^m h_1^{-1} \omega_3 h_2^{-1} \omega_5 \Rightarrow \omega_4 h_2 \underbrace{\omega_1 h}_{\bar h} \omega^l \times \underbrace{h^{-1} \omega_1^{-1}}_{\bar h^{-1}} \underbrace{\omega_1 \omega_2 \omega_3}_{\omega} \underbrace{\omega_3^{-1} h_1}_{\bar h} \omega^m \underbrace{h_1^{-1} \omega_3}_{\bar h^{-1}} h_2^{-1} \omega_5 \Rightarrow \\
& \omega_4 h_2 \bar h \omega^l \bar h^{-1} h_2^{-1} \omega_4^{-1} \omega_4 h_2 \omega h_2^{-1} \omega_5 \omega_5^{-1} \times h_2 h \omega^m h^{-1} h_2^{-1} \omega_5 \Rightarrow \underbrace{\omega_4 h_2 \bar h}_{\alpha} \omega^l \underbrace{\bar h^{-1} h_2^{-1} \omega_4^{-1}}_{\alpha^{-1}} \omega_4 h_2 \omega h_2^{-1} \underbrace{\omega_5 \omega_5^{-1} h_2 h}_{\beta} \omega^m \underbrace{h^{-1} h_2^{-1} \omega_5}_{\beta^{-1}} \Rightarrow \\
& \Rightarrow \alpha \omega^l \alpha^{-1} \omega_4 h_2 \omega h_2^{-1} \omega_5 \beta \omega^m \beta^{-1} \Rightarrow \alpha \omega^l \alpha^{-1} \underbrace{\omega_4 \omega_5}_{\omega} \underbrace{\omega_5^{-1} h_2}_{\gamma} \omega \underbrace{h_2^{-1} \omega_5}_{\gamma^{-1}} \beta \omega^m \beta^{-1} \Rightarrow \\
& \Rightarrow \underbrace{\alpha \omega^l \alpha^{-1}} \omega \underbrace{\gamma \omega \gamma^{-1}} \underbrace{\beta \omega^m \beta^{-1}}
\end{aligned} \tag{$**$}$$

Hence, it is easy to deduce that any combination of form $(*)$ is reduced to form $(**)$, i.e., to a finite product of combinations of the form $\alpha\omega^l\alpha^{-1}$ (#), where l is a positive or negative integral exponent.

Thus, for any element $[\lambda] \in \operatorname{Ker} i_*$, there exists its representative, viz., a line loop λ whose word $\omega(\lambda)$ consists of combinations of the form (#) only. Conversely, it is obvious that if a line loop λ has a word $\omega(\lambda)$ only consisting of combinations of the form (#), then it determines an element from $\operatorname{Ker} i_*$.

Exercise 6°. Prove that the set of words of the described form is a normal subgroup N of the group G and is generated by the element $\omega = \omega(Q)$.

It follows from the computation of the kernel $\operatorname{Ker} i_*$ and Exercise 6 that $\operatorname{Ker} i_* = N$, and therefore $\pi_1(X, x) = G/N$. The latter equality is equivalent to introducing the unique relation $\omega = e$ among the generators of the group G. ■

We now list some corollaries to Theorem 3.

COROLLARY 1. *The fundamental group $\pi_1(RP^2, p)$ of the projective plane RP^2 is a cyclic group Z_2 of order 2.*

PROOF. The surface $X = RP^2$ possesses a canonical development whose word is $a_1 a_1$, therefore (X, x_0) is a cyclic group having one generator a_1 and the defining relation $a_1^2 = e$. ■

COROLLARY 2. *The fundamental group of the torus $\pi_1(T^2, x_0)$ is a free Abelian group with two generators.*

PROOF. The torus T^2 possesses a canonical development with the word $aba^{-1}b^{-1}$, and, consequently, we obtain that the group $\pi_1(T^2, x_0)$ is generated by a, b. The relation $aba^{-1}b^{-1} = e$ provides a condition for its commutativity, viz., $ab = ba$. ∎

Geometrically, to the generator a_1 of the fundamental group of the projective plane, there corresponds its absolute (see the models of RP^2 in Sec. 4, Ch. II). To the generators a_1, b_1 of the fundamental group of the torus T^2, there correspond its parallel and meridian, the two principal noncontractible loops on the torus.

Exercise 7°. Find out what geometric meaning the generators of the fundamental groups have for the surfaces M_p, N_q.

The fundamental group of the knot complement plays an important part in knot classification.

Exercise 8°. Prove that the trivial knot is not equivalent to either the trefoil or figure-of-eight knots.

Hint: Show that the fundamental groups of the complements in R^3 to these knots are not isomorphic.

5. The Topological Invariance of the Euler Characteristic of a Surface.

Let X and X' be two homeomorphic closed surfaces with some subdivisions Π, Π'; let $\chi(\Pi)$ and $\chi(\Pi')$ be their Euler characteristics calculated relative to the subdivisions Π, Π', respectively. We shall prove that $\chi(\Pi) = \chi(\Pi')$.

The development $\Pi(\Pi')$ is reduced to the canonical by equivalent Type I or II transformations (determined by the number of handles $p(p')$ or Möbius strips $q(q')$ that are pasted to the sphere). $2p\,(2p')$, $q\,(q')$ are the numbers of generators (connected by the defining relation $\omega = e$) of the fundamental group of the surface. Owing to the homeomorphism of X and X', the groups $\pi_1(X)$ and $\pi_1(X')$ are isomorphic. Therefore, if the canonical type of the developments Π, Π' corresponds to Type I, then $2p = 2p'$, whence $\chi(\Pi) = \chi(\Pi')$ in virtue of the equalities $\chi(\Pi) = 2 - 2p$ and $\chi(\Pi') = 2 - 2p'$. The case of Type II canonical developments can be considered in an analogous manner. Thus, two different M_p-type surfaces of different genera (this is equally true for the N_q-type surfaces) are not homeomorphic.

Two M_p- and N_q- ($q \geqslant 1$) type surfaces are not homeomorphic either. This follows from the fact that the fundamental groups of an orientable surface M_p of genus p and a nonorientable surface N_q, $q \geqslant 1$, of genus q are not isomorphic. In fact, $\pi_1(M_p)$ is a group with generators $a_1, \ldots, a_p, b_1, \ldots, b_p$ and the defining relation $a_1b_1a_1^{-1}b_1^{-1} \ldots a_pb_pa_p^{-1}b_p^{-1} = e$, whereas the group $\pi_1(N_q)$ is a group with generators $a_1, \ldots, a_q$ and the defining relation $a_1a_1a_2a_2 \ldots a_qa_q = e$. It is clear that these groups are not isomorphic if $2p \neq q$. If we assume, however, that $2p = q$, then $\pi_1(M_p)$ is not isomorphic to $\pi_1(N_q)$ because in the factor group $\pi_1(N_q)/[\pi_1(N_q), \pi_1(N_q)]$ of the group $\pi_1(N_q)$ relative to its commutant $[\pi_1(N_q), \pi(N_q)]$, there is a coset of order 2, which contains the element $a_1a_2 \ldots a_q$. There are no elements of the second order in the factor group $\pi_1(M_p)/[\pi_1(M_p), \pi_1(M_p)]$, since the element $a_1b_1a_1^{-1}b_1^{-1} \ldots a_pb_pa_p^{-1}b_p^{-1}$ of the free group with the generators $a_1, \ldots, a_p, b_1 \ldots, b_p$ is contained in the commutant of this group.

Note that the classification Theorem 2 of Sec. 4, Ch. II, is now proved completely. The genus of a surface and its orientability or nonorientability completely determine its topological type.

6. On Calculating Higher Homotopy Groups.

The calculation of the homotopy groups of spaces is an important but a difficult problem. Methods for calculating these groups have been worked out. However, when they are applied to concrete cases, considerable difficulties are encountered. Nevertheless, certain homotopy groups have been calculated for sufficiently 'good' spaces and play an important part in many problems.

The following theorem enables us to reduce the computation of the homotopy groups of a space X to that of the corresponding groups of a space Y which is homotopy equivalent to X.

THEOREM 4. *If $f : X \to Y$ is a homotopy equivalence then for any point $x \in X$, the homomorphism*

$$f_n : \pi_n(X, x) \to \pi_n(Y, f(x))$$

which is induced by f, is an isomorphism.

PROOF. Let a mapping g be homotopy inverse of f, and φ a representative of a certain class $[\varphi] \in \pi_n(X, x_0)$. Then $(gf)\varphi$ is a representative of its image $(gf)_n[\varphi]$. The spheroid φ is 'attached' to a point x_0, and the spheroid $(gf)\varphi$ to the point $(gf)(x_0) = z_0$, the former being homotopic to the latter in virtue of $gf \sim 1_X$. Suppose that the point x_0 shifts to a point z_0 under this homotopy, describing a path $\omega(t)$ in doing so (Fig. 71).

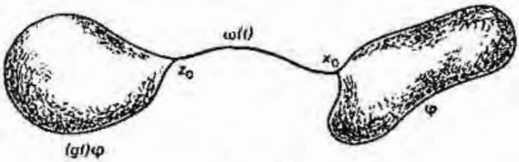

Fig. 71

Let $\omega(t)$ induce an isomorphic mapping $\pi_n(X, z_0) \xrightarrow{S_n^{\omega}} \pi_n(X, x_0)$ (see Theorem 5, Sec. 3). The homotopy of the spheroids φ and $(gf)\varphi$ generates the homotopy of the spheroids $gf\varphi$ and α from $S_n^{\omega^{-1}}[\varphi]$. Therefore, $g_n f_n[\varphi] = [gf\varphi] = [\alpha] = S_n^{\omega^{-1}}[\varphi]$, which means that the following diagram is commutative:

$$\begin{tikzcd}
\pi_n(X, x_0) \arrow[r, "f_n"] \arrow[d, "S_n^{\omega^{-1}}"'] & \pi_n(Y, f(x_0)) \arrow[dl, "g_n"] \\
\pi_n(X, gf(x_0)) &
\end{tikzcd}$$

Similarly, it may be shown (try it yourself) that the diagram

$$\begin{tikzcd}
 & \pi_n(Y, f(x_0)) \arrow[dl, "g_n"'] \arrow[dr, "S_n^{(\omega')^{-1}}"] & \\
\pi_n(X, gf(x_0)) \arrow[rr, "f_n"] & & \pi_n(Y, (fgf)(x_0))
\end{tikzcd}$$

is commutative, where $\omega' : I \to Y$ is a path from the point $f(x_0)$ to the point $(fgf)(x_0)$, equal to $f\omega$. It follows from the commutativity of these diagrams (taking into account that $S_n^{\omega-1}$, $S_n^{(\omega')-1}$ are isomorphisms) that f_n, g_n are isomorphisms. ◻

Exercises.

9°. Prove that the one-point space and the circumference S^1 have different homotopy types.

10°. Prove that the two-dimensional disc and the two-dimensional cylinder over a circumference have different homotopy types.

Although it has not yet been completely resolved at present, the problem of calculating the groups $\pi_k(S^n)$ has stimulated the development of many branches of modern topology. Here are two very distinct cases, viz., $k \leqslant n$ and $k > n$. Though requiring a development of a special method, the former is elementary enough. We list the following results without proof*:

$$\pi_1(S^n) = \pi_2(S^n) = \ldots = \pi_{n-1}(S^n) = 0, \pi_n(S^n) = Z (n \geqslant 1).$$

Hence, it follows, in particular, that the sphere S^n is noncontractible to any of its points.

The second case has not been fully investigated, and the difficulties increase with the growth of n and $k - n$. Here are some of the simplest results:

$$\pi_3(S^2) = Z, \pi_4(S^3) = Z_2, \ldots, \pi_{n+1}(S^n) = Z_2 (n \geqslant 3).$$

This refutes the intuitive assumption that $\pi_k(S^n) = 0$ when $k > n$.

Thus, when $n = 1, 2, \ldots$ the groups $\pi_n(S^n)$ are free Abelian groups with one generator γ, γ_n being the homotopy class of the identity mapping $1_{S^n} : S^n \to S^n$. The multiple classes $l \cdot \gamma_n$ can be imagined as the homotopy classes of mappings $\omega: S^n \to S^n$ such that 'twist' the sphere S^n onto itself l times. In addition, if $l > 0$, then the orientation of the sphere under the mapping φ is said to be preserved, whilst if $l < 0$, the orientation is said to be changed (cf. the homotopy classes from $\pi_1(S^1)$).

Exercise 11°. Let S^n be a sphere with the centre at the origin of the space R^{n+1}. Show that the mapping of S^n into itself given by the correspondence

$$(x_1, x_2, \ldots, x_{n+1}) \to (-x_1, x_2, \ldots, x_{n+1}),$$

determines a homotopy class equal to $(-\gamma_n)$.

It is quite simple to prove that $\pi_n(X, x_0) = 0$, $n \geqslant 1$, if the space X is contractible to a point (Theorem 4 should be applied). In particular, we obtain the following

* The group $\pi_1(S^1)$ was calculated in the proof of Theorem 1.

for the disc D^n and space R^n:

$$\pi_k(\bar{D}^n) = 0,\ \pi_k(D^n) = 0,\ \pi_k(R^n) = 0,\ k = 1, 2, \ldots.$$

7. Some Applications.

We first prove an important property of the sphere S^n.

THEOREM 5. *The sphere S^n (i.e., the boundary of the disc $\bar{D}^{n+1}$) is not a retract of $\bar{D}^{n+1}$*

PROOF. A necessary condition for the existence of an extension of a mapping was indicated in Sec. 2. It was formulated in terms of a functor to the category of groups. We apply this conditions, taking the functor π_n as the functor T. We know already that $\pi_n(S^n) = Z$, $\pi_n(\bar{D}^{n+1}) = 0$, and if the sphere S^n were a retract of $\bar{D}^{n+1}$, then we would obtain the following commutative diagram

$$\begin{tikzcd} \bar{D}^{n+1} \arrow[dr, "r"'] & & S^n \arrow[ll, "i"'] \arrow[dl, "1_{S^n}"] \\ & S^n & \end{tikzcd} \ (1) \qquad \begin{tikzcd} \pi_n(\bar{D}^{n+1}) \arrow[dr, "r_*"'] & & \pi_n(S^n) \arrow[ll, "i_*"'] \arrow[dl, "1_{\pi_n(S^n)}"] \\ & \pi_n(S^n) & \end{tikzcd} \ (2) \qquad \begin{tikzcd} 0 \arrow[dr, "r_*"'] & & Z \arrow[ll, "i_*"'] \arrow[dl, "1_Z"] \\ & Z & \end{tikzcd} \ (3)$$

where i is the embedding of the sphere into the ball, and r the required retraction. Since π_n is a covariant functor, it would convert diagram (1) into the commutative diagram (2) which is of form (3). The latter is contrary to its commutativity. Therefore, the assumption that there is a retraction r is not valid.

With the aid of Theorem 5, the following interesting theorem, which has important applications, is proved.

THE FIXED-POINT THEOREM (BROUWER). *Any continuous mapping $f : \bar{D}^{n+1} \to \bar{D}^{n+1}$ of an $(n + 1)$-dimensional closed ball (disc) into itself possesses at least one fixed point, i.e., there exists a point $x_* \in \bar{D}^{n+1}$ such that $f(x_*) = x_*$.*

PROOF. In fact, if there is no such point, i.e., for any point $x \in D^{n+1}$, $f(x) \neq x$, then the line-segment joining the point $f(x)$ to the point x can be extended beyond the point x to meet the sphere S^n at a certain point $r(x)$. Then the mapping $r : \bar{D}^{n+1} \to S^n$, $x \to r(x)$ is an extension of the identity mapping of the sphere S^n to $\bar{D}^{n+1}$. But we have just proved that there is no such extension. The contradiction proves the theorem. ∎

8. The Degree of a Mapping.

The group $\pi_n(S^n) = Z$ is closely related to the notion of the degree of a continuous mapping $f : S^n \to S^n$, which is often used in analysis. Let γ_n be a generator of the group $\pi_n(S^n)$. Then $f_*(\gamma_n) = \alpha\gamma_n$, where α is an integer, and f_* a homomorphism of the group $\pi_n(S^n)$ induced by the mapping f. The number α is called the *degree of the mapping* f and denoted by $\deg f$ (the sign of $\deg f$ does not depend on the choice of a generator).

Exercises.

12°. The mapping of the unit circumference $S^1 = \{z : |z| = 1\}$ of the complex plane is given by the formula $f(z) = z^n$. Show that $\deg f = n$.

13°. Show that if $f : S^1 \to S^1$ is a local homeomorphism, then the number of points in the full inverse image $f^{-1}(x)$ of any point $x \in S^1$ is constant and equal to $|\deg f|$.

The notion of the degree of a mapping is also introduced naturally for the mappings $f : S_1^n \to S_2^n$ from one replica of the sphere to another. (To do this, the basis

classes γ_n^1 in $\pi_n(S_1^n)$ and γ_n^2 in $\pi_n(S_2^n)$ should be fixed, and then $f_*(\gamma_n^1) = \deg f \cdot \gamma_n^2$.) Since γ_n is the homotopy class $[1_{S^n}]$ of the identity mapping, we have the following relation for the mapping $f : S^n \to S^n$:

$$f_*(\gamma_n) = f_*[1_{S^n}] = [f1_{S^n}] = [f],$$

therefore, $\deg f \cdot \gamma_n$ is the homotopy class of the mapping f; thus, $\deg f$ is 'the number' of the homotopy class $[f]$.

If $f = I$ is the identity mapping, then $\deg f = 1$; if $f \sim 0$ (homotopic to the constant mapping) then $\deg f = 0$; if $f : S^n \to S^n$, $g : S^n \to S^n$ are two mappings, then they are homotopic if and only if they are of the same degree: $\deg f = \deg g$. We also adduce one useful formula, viz., $\deg(fg) = (\deg f) \cdot (\deg g)$ that follows from the relation $[fg] = f_*[g]$.

The notion of degree is used while investigating whether it is possible to extend continuous mappings $f : S^n \to R^{n+1} \setminus \{0\}$ to the ball $\bar{D}^{n+1}$ bounded by the sphere S^n. Since the space $R^{n+1} \setminus \{0\}$ is homotopy equivalent to S^n, their homotopy groups are isomorphic and therefore we may speak of the degree of a given mapping, usually called the *characteristic* (or *rotation*) *of the vector field* f; we denote it by $\chi_{S^n}(f)$.

LEMMA 6. *The condition $\chi_{S^n}(f) = 0$ is necessary and sufficient for the extension $\bar{f} : \bar{D}^{n+1} \to R^{n+1} \setminus \{0\}$ of the mapping f to exist.*

The proof is evident from our note that the mapping $\bar{f}$ determines the homotopy $f \sim 0$ by the formula

$$f(x, t) = \bar{f}(tx), x \in S^n, t \in [0, 1]$$

(if S^n is the sphere of radius 1 and centre at 0), and vice versa.

Exercise 14°. Construct the extension $\bar{f}$ when $f \sim 0$.

An obvious corollary follows from Lemma 6.

COROLLARY. *If $\chi_{S^n}(f) \neq 0$ then any extension $\bar{f} : \bar{D}^{n+1} \to R^n$ has a zero, i.e., there exists a point $x_0 \in D^{n+1}$, $\bar{f}(x_0) = 0$.*

This corollary is often used for the proof of the existence of a solution to the equation $\bar{f}(x) = 0$, where $\bar{f} : \bar{D}^{n+1} \to R^{n+1}$ is a given mapping.

EXAMPLES.

1. It is easy to verify that with the conditions of the Brouwer fixed-point theorem, the mapping $\bar{f}(x) = -f(x) + x$ either has a zero on S^n or $\chi_{S^n}(\bar{f}) = 1(\bar{f} : S^n \to R^{n+1} \setminus \{0\}$ is homotopized to the identity mapping $\bar{f}(x, t) = -tf(x) + x, x \in S^n$, $0 \leqslant t \leqslant 1)$. Therefore, $\bar{f}$ has a zero in $\bar{D}^{n+1}$.
2. The fundamental theorem of algebra: a complex polynomial

$$\bar{f}(z) = z^m + a_1 z^{m-1} + \ldots + a_{m-1} z + a_m$$

has a root in the complex plane.

We denote the circumference on the z-plane $\{z : |z| = \rho\}$ by S_ρ^1.

LEMMA 7. *For a sufficiently large ρ, we have*

$$\bar{f} : S_\rho^1 \to R^2 \setminus \{0\},$$

and in addition $\chi_{S^1_\rho}(\tilde{f}) = m$.

PROOF. Consider the homotopy

$$\tilde{f}(z, t) = z^m + t(a_1 z^{m-1} + \ldots + a_{m-1} z + a_m), \ t \in [0, 1].$$

We have an estimate

$$|\tilde{f}(z, t)| \geqslant |z|^m \left[1 - t\left(a_1 \frac{1}{|z|} + \ldots + a_{m-1} \frac{1}{|z|^{m-1}} + a_m \frac{1}{|z|^m}\right)\right], \ |z| \neq 0.$$

It is evident that there is $\rho > 0$, sufficiently large for $|\tilde{f}(z, t)| > 0$ when $|z| = \rho$, $t \in [0, 1]$. Therefore $\tilde{f} : S^1_\rho \to R^2 \setminus \{0\}$ is homotopic to the mapping $g : S^1_\rho \to R^2 \setminus \{0\}$, $g(z) = z^m$. According to Exercise 12, $\chi_{S^1_\rho}(g) = m$; therefore $\chi_{S^1_\rho}(f) = m$ as well. To complete the proof, we now use the corollary to Lemma 6. ■

FURTHER READING

As regards the classical topics of homotopy theory covered in this chapter, we recommend, above all, *Homotopy Theory* by Hu Sze-tsen [43], as it provides a thorough and quite complete account of homotopy theory. In addition, the reader will find it useful to see the accounts of these topics (in some instances, somewhat formalized, but quite detailed) in Spanier's *Algebraic Topology* [73]. We also recommend *Homotopy Theory* [33] by Fuchs et al. and *Modern Geometry* [28] by Dubrovin et al. as both of them are noteable for their geometrical approach. The elements of homotopy theory can be found in *First Course of Topology. Geometric Chapters* [70] by Rohlin and Fuchs, which contains a systematic approach to homotopy group theory. A good book of problems related to this chapter is *Problems in Geometry* [61] by Novikov et al.

To study the notions of category and functor, see Maclane's *Homology* [51] (Ch. I, Secs. 7 and 8). These topics are also well covered in Spanier's *Algebraic Topology* [73] (Ch. 1, Secs 1 and 2).

A considerable part of the book by Massey *Algebraic Topology: An Introduction* [52] is devoted to fundamental groups and related topics, Chs. 2 and 3 being recommended for an initial study of these topics.

In our proof of the theorem about the fundamental group of a surface, we quite closely follow *Lehrbuch der Topologie* [71] (Ch. VII) by Seifert and Threlfall. The theory of the fundamental group is also expounded visually and fully by Dubrovin et al. in *Modern Geometry* [28].

A strict topological theory of the degree of a mapping and the vector field characteristic (based on homology theory) may be found in *Combinatorial Topology* [1] (Ch. XVI, Secs. 3 and 5) by Alexandrov. Some applications of the theory of the degree of a mapping are given in *Geometric Methods of Nonlinear Analysis* [47] by Krasnoselsky and Zabreiko.

IV

Manifolds and Fibre Bundles

We considered the general properties of topological spaces and their mappings in the previous chapters. However, there are spaces with other structures in topology and its applications, e.g., smooth manifolds and fibre spaces,which play an important part in many branches of modern mathematics. In this chapter,we study smooth manifolds in detail as well as tangent bundles which are naturally related to them. We cover the elements of the theory of critical points of smooth functions on manifolds and deal with the elements of fibre space theory.

1. BASIC NOTIONS OF DIFFERENTIAL CALCULUS IN n-DIMENSIONAL SPACE

1. Smooth Mappings. Remember that R^n is the space of ordered sets $x = (x_1, \ldots, x_n)$ of n real numbers (see Sec. 2, Ch. II) called *points* or *vectors*. We will assume that R^n is *standardly embedded* in R^{n+k}, i.e., a point $(x_1, \ldots, x_n)$ from R^n will be identified with the point $(x_1, \ldots, x_n, 0, \ldots, 0)$ from R^{n+k}. The numbers $x_1, \ldots, x_n$ from the set $(x_1, \ldots, x_n)$ are called the *standard coordinates of the point* $x = (x_1, \ldots, x_n)$ in R^n.

Let $U \subset R^n$ be an open set. Any mapping $f: U \to R^m$ can be represented (see Sec. 2, Ch. II) as an ordered set of m functions:

$$f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), f_2(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)).$$

DEFINITION 1. A mapping $f: U \to R^m$ is said to be *smooth* (or *differentiable*) of class C^r, $r \geqslant 1$, on U if each function f_k, $k = 1, \ldots, m$, has all continuous partial derivatives $\dfrac{\partial^s f_k}{\partial^{s_1} x_1 \ldots \partial^{s_n} x_n}$, $s_1 + \ldots + s_n = s$, on U for every order up to $s = r$ inclusive.

Smooth mappings f of class C^r are also called C^r-*mappings* and written as $f \in C^r$.

If all the functions f_k possess continuous partial derivatives of any order then the mapping f is said to be *infinitely smooth* ($f \in C^\infty$). Continuous mappings are called C^0-*mappings*. It is obvious that the following relations are valid

$$C^0 \supset C^1 \supset \ldots \supset C^r \supset \ldots \supset C^\infty.$$

In case all the functions f_k are analytic (a function is said to be *analytic* if its Taylor expansion converges to it in the neighbourhood of each point), the mapping f is said to be *analytic* ($f \in C^\omega$). The following set inclusion is valid $C^\infty \supset C^\omega$.

DEFINITION 2. The matrix

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} \cdots \dfrac{\partial f_1}{\partial x_n} \\ \cdots\cdots\cdots \\ \dfrac{\partial f_m}{\partial x_1} \cdots \dfrac{\partial f_m}{\partial x_n} \end{pmatrix}$$

of the first derivatives of the mapping f, calculated at a point x_0 is called *the Jacobian matrix of the mapping* f at x_0 and denoted by $\left(\dfrac{\partial f}{\partial x}\right)\Big|_{x_0}$

The Jacobian matrix determines a linear mapping $R^n \to R^m$:

$$\begin{cases} y_1 = \left.\dfrac{\partial f_1}{\partial x_1}\right|_{x_0} x_1 + \left.\dfrac{\partial f_1}{\partial x_2}\right|_{x_0} x_2 + \ldots + \left.\dfrac{\partial f_1}{\partial x_n}\right|_{x_0} x_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_m = \left.\dfrac{\partial f_m}{\partial x_1}\right|_{x_0} x_1 + \left.\dfrac{\partial f_m}{\partial x_2}\right|_{x_0} x_2 + \ldots + \left.\dfrac{\partial f_m}{\partial x_n}\right|_{x_0} x_n \end{cases}$$

which is called the *derivative of the mapping* f at the point x_0 and denoted by $D_{x_0}f$. The derivative is a 'linearization' of the mapping f, i.e., the affine mapping $f(x_0) + (D_{x_0}f)(x - x_0)$ coincides with $f(x)$ up to infinitesimals of a higher order than $\|x - x_0\|$. More precisely, $D_{x_0}f$ is a unique linear mapping of R^n to R^m for which

$$\frac{f(x) - f(x_0) - (Dx_0f)(x - x_0)}{\|x - x_0\|} \to 0 \quad \text{as} \quad x \to x_0.$$

A corollary to the theorem about the derivative of a composite function is the following statement (the 'chain rule'): *under a superposition of mappings f and g, their Jacobian matrices are multiplied together, i.e.,*

$$\left.\left(\frac{\partial(fg)}{\partial x}\right)\right|_{x_0} = \left.\left(\frac{\partial f}{\partial y}\right)\right|_{g(x_0)} \left.\left(\frac{\partial g}{\partial x}\right)\right|_{x_0}.$$

THE PROOF of this is left to the reader as an exercise.

2. The Rank of a Mapping. Let $U \subset R^n$ be an open set, $f: U \to R^m$ a mapping of class $C^r (r \geqslant 1)$. The *rank of the mapping* f at a point x_0 is the rank of its Jacobian matrix calculated at the point x_0 and denoted by rank $_{x_0}f$. It equals the dimension of the subspace in R^m, viz., the image of R^n under the linear mapping $D_{x_0}f$. Since the rank of a matrix cannot exceed the number of rows or columns, we must have rank $_{x_0}f \leqslant \min(n, m)$. Points at which rank $_{x_0}f = \min(n, m)$ are said to be *regular* (also *noncritical* or *nonsingular*), and points at which rank $_{x_0}f <$ $< \min(n, m)$ are said to be *nonregular* (also *critical* or *singular*).

The set of regular points of a mapping f is open in R^n (owing to the continuity of the partial derivatives, the determinant for which the rank of the Jacobian matrix is found is different from zero in some neighbourhood of a regular point). The set of regular points may be empty (give some examples).

3. The Implicit Function Theorem. We now adduce the theorem about implicit functions that is proved in the course of analysis. We shall represent points of the space $R^{n+m} = R^n \times R^m$ in the form (x, y), where $x = (x_1, \ldots, x_n) \in R^n$, $y = (y_1, \ldots, y_m) \in R^m$. Let $U \subset R^n$, $V \subset R^m$ be open sets, and $(x_0, y_0) \in U \times V \subset R^{n+m}$

THEOREM 1. *If $f: U \times V \to R^m$ is a C^1-mapping, $f(x_0, y_0) = 0$ and $\det \left.\left(\dfrac{\partial f}{\partial y}\right)\right|_{(x_0, y_0)} \neq 0$, then there exists an open neighbourhood $W(x_0) \subset U$ of the point x_0 and a mapping $g: W(x_0) \to V$ such that $g(x_0) = y_0$ and $f(x, g(x)) = 0$ for*

any $x \in W(x_0)$, the mapping g being unique. Furthermore, $g \in C^1$ and

$$\left(\frac{\partial g}{\partial x}\right) = -B^{-1}A, \tag{1}$$

where the matrices B and A are obtained from the matrices

$$\left(\frac{\partial f}{\partial y}(x, y)\right) \text{ and } \left(\frac{\partial f}{\partial x}(x, y)\right),$$

respectively, on replacing the argument y by $g(x)$.

NOTE. If $f \in C^r$, $r \geq 1$, then $g \in C^r$. This statement follows from equality (1).

One corollary to the implicit function theorem is the inverse mapping theorem.

THEOREM 2. *Let $U \subset R^n$ be an open set, $f: U \to R^n$ a mapping of class C^r, $r \geq 1$, $x_0 \in U$ a regular point of the mapping f. Then there exist open neighbourhoods $V(x_0)$, $W(f(x_0))$ of the points x_0 and $f(x_0)$ such that f is a homeomorphism $V(x_0) \xrightarrow{f} W(f(x_0))$ and $f^{-1} \in C^r$.*

PROOF. Consider the mapping $F: R^n \times U \to R^n$ given by the rule $F(y, x) = y - f(x)(y \in R^n,\ x \in U)$. Denote $y_0 = f(x_0)$. It is obvious that $F \in C^r$ and $F(y_0, x_0) = 0$. Since $\operatorname{rank}_{x_0} f = n$, it follows that $\det\left(\frac{\partial F}{\partial x}\right)\Big|_{(y_0, x_0)} \neq 0$. By the implicit function theorem,there exists an open neighbourhood $W(y_0) \subset R^n$ of the point y_0 and a unique mapping $g: W(y_0) \to U$ such that $g(y_0) = x_0$ and for any $y \in W(y_0)$,

$$F(y, g(y)) = y - f(g(y)) = 0. \tag{2}$$

Put $V(x_0) = g(W(y_0))$. Since $V(x_0) = f^{-1}(W(y_0))$, we find that $V(x_0)$ is open in virtue of the continuity of f. Thus, $g: W(y_0) \to V(x_0)$ is a mapping of open sets and, due to (2), we obtain $g = f^{-1}$. Moreover, from the note to Theorem 1, we have $f^{-1} \in C^r$. ∎

DEFINITION 3. A mapping $f: U \to V$ of an open set $U \subset R^n$ onto an open set $V \subset R^n$ is called a *C^r-diffeomorphism*, $r \geq 1$, if (i) f is a homeomorphism of U onto V, (ii) $f \in C^r$ and (iii) $f^{-1} \in C^r$.

Exercise 1°. Construct the C^∞-diffeomorphism $f: D^n_\rho(x) \to R^n$.

The inverse mapping theorem may now be formulated as follows. *If $f: U \to R^n$ is a C^r-mapping, $r \geq 1$, of an open set $U \subset R^n$ in R^n, and x_0 is a regular point of f, then there exist open neighbourhoods $V(x_0)$, $W(f(x_0))$ of the points x_0, $f(x_0)$ such that the mapping $f: V(x_0) \to W(f(x_0))$ is a C^r-diffeomorphism.*

Exercises.

2°. Prove that a diffeomorphism has no nonregular points.

Hint: Use the note about the Jacobian matrix for the superposition of mappings f, f^{-1}.

3°. Let $f: R^n \to R^1$ be a mapping of class C^r, $r \geq 1$. Show that the nonregular

points of f are characterized by the fact that the first partial derivatives of f vanish at them for all variables.

4. 'Curvilinear' Coordinate Systems.

Let $U, V \subset R^n$ be open sets, and $f : U \to V$ a homeomorphism. The position of every point $y \in V$ may be specified by means of the standard coordinates $y_1, \ldots, y_n$ of the point y, but this can also be done with the help of the standard coordinates of the point $x = f^{-1}(y) \in U$.

DEFINITION 4. The standard coordinates of the point $f^{-1}(y) \in U$ are termed the '*curvilinear*' *coordinates* of the point $y \in V$.

In other words, instead of the coordinate planes $y_i = b_i$, $i = 1, \ldots, n$, we consider in V the images, under the homeomorphism f, of the coordinate planes $x_i = a_i$, $i = 1, \ldots, n$, in U, the intersection of these images determining the position of the point y. The term 'curvilinear' coordinates merely reflects the fact that the new coordinate 'planes' in V are, generally speaking, 'curvilinear' (Fig. 72).

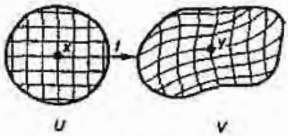

Fig. 72

Note that in analysis, curvilinear coordinates are not usually introduced by means of a homeomorphism, but by using a C^r-diffeomorphism, where the order of smoothness r depends on the problem under consideration.

If a function g is given on V as a function of the standard coordinates of a point y, then it can be considered as a function of the standard coordinates of a point x, i.e., as a function of the curvilinear coordinates of the point y. In analysis, such an operation is termed a *change of variables*. In other words, we replace the coordinates in the inverse image space of the function g, which is equivalent to considering the function gf instead of g. It goes without saying that mappings and similar changes of variables can also be considered. These changes are also made in the space of mapping images, i.e., if $g : W \to V$ is a mapping of a set $W \subset R^m$ then instead of the standard coordinates of the point $g(z)$, $z \in W$, the curvilinear coordinates of the point $g(z)$ determined by the homeomorphism f are considered. Such a change is equivalent to considering the mapping $f^{-1}g$ instead of a mapping g.

Note that the rank of a smooth mapping is unaltered under a smooth change of variables.

5. A Theorem on Rectifying.

The *standard embedding* of R^n into R^{n+k} is a mapping $R^n \to R^{n+k}$ specified by the correspondence

$$(x_1, \ldots, x_n) \to (x_1, \ldots, x_n, 0, \ldots, 0).$$

The *standard projection* of R^{n+k} onto R^n is the mapping $R^{n+k} \to R^n$ determined by the correspondence $(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}) \to (x_1, \ldots, x_n)$.

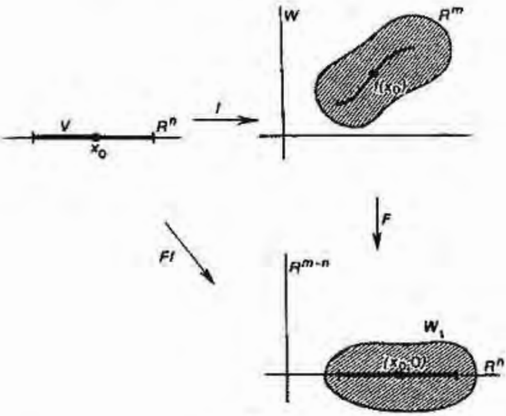

Fig. 73

THEOREM 3 (ON RECTIFYING A MAPPING IN THE NEIGHBOURHOOD OF A REGULAR POINT). *Let $U \subset R^n$ be an open set, $f: U \to R^m$ a C^r-mapping, $r \geqslant 1$ and x_0 a regular point of f.*

(A) *If $n \leqslant m$ then there exist an open neighbourhood $W(f(x_0))$ of the point $f(x_0)$, an open set $W_1 \subset R^m$, and a C^r-diffeomorphism $F: W(f(x_0)) \to W_1$ such that Ff on a certain open neighbourhood $V(x_0) \subset R^n$ is the standard embedding of R^n into R^m* (Fig. 73).

(B) *If $n \geqslant m$ then there exist an open neighbourhood $V(x_0)$ of the point x_0, an open set $W \subset R^n$, and a C^r-diffeomorphism $F: V(x_0) \to W$ such that fF^{-1} on the set W is the standard projection of R^n onto R^m* (Fig. 74).

Consider the meaning of this theorem from the point of view of coordinate changes. In case (A), the diffeomorphism F^{-1} determines the curvilinear coordinates $\{\xi_1, \ldots, \xi_m\}$ in the space R^m, in whose terms the mapping f is of the form $\xi_1 = x_1, \ldots, \xi_n = x_n, \xi_{n+1} = 0, \ldots, \xi_m = 0$.

In case (B), the diffeomorphism F^{-1} determines the curvilinear coordinates $\{\xi_1, \ldots, \xi_n\}$ in the space R^n, in whose terms the mapping f is of the form $y_1 = \xi_1, \ldots, y_m = \xi_m$.

The idea of the proof is to complete the given mappings to mappings of spaces of the same dimensions, and, by applying the inverse mapping theorem, to obtain the necessary coordinate changes.

PROOF. (A) Represent points of the space $R^m = R^n \times R^{m-n}$ in the form (x, y), where $x = (x_1, \ldots, x_n) \in R^n$, $y = (y_1, \ldots, y_{m-n}) \in R^{m-n}$. Given that rank ${}_{x_0}f = n$, or rank $\left(\frac{\partial f}{\partial x}\right)\bigg|_{x_0} = n$, we assume, at first, that the determinant made up

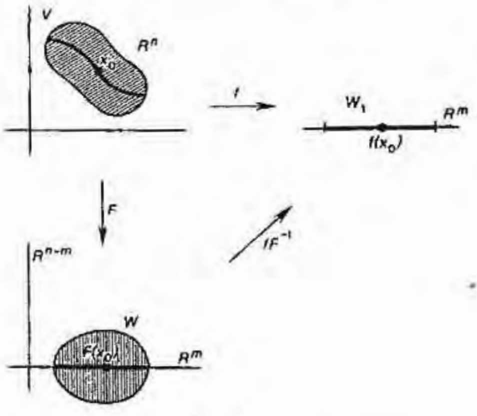

Fig. 74

of the first n rows of the Jacobian matrix $\left(\dfrac{\partial f}{\partial x}\right)\Big|_{x_0}$ is different from zero. Consider the mapping $F_1 : U \times R^{m-n} \to R^m$ given by the rule $F_1(x, y) = f(x) + (0, y)$. The Jacobian matrix of the mapping F_1 at the point $(x_0, 0)$ is of the form

$$\left(\begin{array}{ccc|ccc}
\dfrac{\partial f_1}{\partial x_1}\Big|_{x_0}, & \ldots, & \dfrac{\partial f_1}{\partial x_n}\Big|_{x_0} & & & \\
\ldots & \ldots & \ldots & & 0 & \\
\dfrac{\partial f_n}{\partial x_1}\Big|_{x_0}, & \ldots, & \dfrac{\partial f_n}{\partial x_n}\Big|_{x_0} & & & \\
\hline
\dfrac{\partial f_{n+1}}{\partial x_1}\Big|_{x_0}, & \ldots, & \dfrac{\partial f_{n+1}}{\partial x_n}\Big|_{x_0} & 1 & & 0 \\
\ldots & \ldots & \ldots & & \ddots & \\
\dfrac{\partial f_m}{\partial x_1}\Big|_{x_0}, & \ldots, & \dfrac{\partial f_m}{\partial x_n}\Big|_{x_0} & 0 & & 1
\end{array}\right)$$

By the assumption, the determinant in the upper left-hand corner is other than zero, therefore rank ${}_{(x_0, 0)}F_1 = m$. By the inverse mapping theorem, there exist open neighbourhoods $W(x_0, 0)$ and $W_1(f(x_0))$ of the points $(x_0, 0)$ and $f(x_0)$, respectively,

such that the mapping $F_1|_{W(x_0, 0)} : W(x_0, 0) \to W_1(f(x_0))$ is a C^r-diffeomorphism. Therefore, the mapping $F_1^{-1} : W_1(f(x_0)) \to W(x_0, 0)$ is also a C^r-diffeomorphism. Put $F = F_1^{-1}$. In virtue of the continuity of the mapping f, there exists an open neighbourhood $V(x_0)$ of the point $x_0 \in R^n$ such that $f(V(x_0)) \subset W_1(f(x_0))$. Then the mapping $Ff : V(x_0) \to W(x_0, 0)$ is valid. The mapping Ff is the standard embedding of R^n into R^m on the neighbourhood $V(x_0)$. In fact, since the mapping F_1^{-1} is bijective and $F_1(x, 0) = f(x)$, we have $(Ff)(x) = F(f(x)) = F_1^{-1}(f(x)) = (x, 0)$.

In case the determinant of the first n rows of the Jacobian matrix $\left(\dfrac{\partial f}{\partial x}\right)\Big|_{x_0}$ is equal to zero, it is first necessary to renumber the coordinates in R^m (in other words, to make a special change of coordinates in R^m by means of the C^∞-diffeomorphism $g : R^m \to R^m$ so that the determinant made up of the first n rows of the Jacobian matrix $\left(\dfrac{\partial (g^{-1}f)}{\partial x}\right)\Big|_{x_0}$ may be different from zero). We construct the C^r-diffeomorphism F for the mapping $g^{-1}f$ as above. Thus, Fg^{-1} will be the required C^r-diffeomorphism for the mapping f.

(B) We represent the elements of the space $R^n = R^m \times R^{n-m}$ in the form (x, y), where $x = (x_1, \ldots, x_m) \in R^m$, $y = (y_1, \ldots, y_{n-m}) \in R^{n-m}$. Let $x_0 = (x^0, y^0)$. From the data, we have $\operatorname{rank}_{(x^0, y^0)} f = m$, or $\operatorname{rank} \left(\dfrac{\partial f}{\partial (x, y)}\right)\Big|_{(x^0, y^0)} = m$. Assuming at first that the determinant made up of the first m columns of the Jacobian matrix $\left(\dfrac{\partial f}{\partial (x, y)}\right)\Big|_{(x^0, y^0)}$ is different from zero, consider the mapping $F : U \to R^m \times R^{n-m}$ given by the rule $F(x, y) = (f(x, y), y)$. The Jacobian matrix of the mapping F at the point (x^0, y^0) is of the form

$$\left(\begin{array}{ccc|ccc}
\dfrac{\partial f_1}{\partial x_1}\Big|_{(x^0, y^0)}, & \ldots, & \dfrac{\partial f_1}{\partial x_m}\Big|_{(x^0, y^0)} & \dfrac{\partial f_1}{\partial y_1}\Big|_{(x^0, y^0)}, & \ldots, & \dfrac{\partial f_1}{\partial y_{n-m}}\Big|_{(x^0, y^0)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\dfrac{\partial f_m}{\partial x_1}\Big|_{(x^0, y^0)}, & \ldots, & \dfrac{\partial f_m}{\partial x_m}\Big|_{(x^0, y^0)} & \dfrac{\partial f_m}{\partial y_1}\Big|_{(x^0, y^0)}, & \ldots, & \dfrac{\partial f_m}{\partial y_{n-m}}\Big|_{(x^0, y^0)} \\
\hline
 & & & 1 & & 0 \\
 & 0 & & & \ddots & \\
 & & & 0 & & 1
\end{array}\right)$$

By the assumption, the determinant in the upper left-hand corner is different from zero, therefore $\operatorname{rank}_{(x^0, y^0)} F = n$. By the inverse mapping theorem there exist open neighbourhoods $V(x^0, y^0)$ and $W(F(x^0, y^0))$ of the points (x^0, y^0) and $F(x^0, y^0)$, respectively, such that the mapping

$$F\mid_{V(x^0, y^0)} : V(x^0, y^0) \to W(f(x^0, y^0), y^0)$$

is a C^r-diffeomorphism. The mapping fF^{-1} on the neighbourhood $W(f(x^0, y^0), y^0)$ is the standard projection of R^n onto R^m. In fact, let $z \in W(f(x^0, y^0), y^0)$. Since F^{-1} is bijective, there is a unique point (ξ, η) in the neighbourhood $V(x^0, y^0)$, for which $z = (f(\xi, \eta), \eta)$, and we find that

$$[fF^{-1}](f(\xi, \eta), \eta) = f[F^{-1}(f(\xi, \eta), \eta)] = f(\xi, \eta).$$

In case the determinant made up of the first m columns of the Jacobian matrix $\left(\frac{\partial f}{\partial (x, y)}\right)\Big|_{(x^0, y^0)}$ equals zero, it is first necessary to renumber the coordinates in R^n (i.e., to change the coordinates by means of a C^∞-diffeomorphism $g : R^n \to R^n$) so that the determinant made up of the first m columns of the Jacobian matrix $\left(\frac{\partial (fg)}{\partial (x, y)}\right)\Big|_{(x^0, y^0)}$ may be different from zero. We construct a C^r-diffeomorphism F for the mapping fg in the manner described above, and then Fg^{-1} is the required C^r-diffeomorphism for the mapping f. ■

6. Lemma about a Representation of Smooth Functions. We adduce here another result which is necessary before we can go any further.

LEMMA 1. *Let f be a C^{r+1}-function ($r \geq 0$) given on a convex neighbourhood $V(x^0)$ of a point x^0 in R^n. Then there exist C^r-functions $g_i : V(x^0) \to R^1$, $i = 1, \dots, n$, such that $f(x) = f(x^0) + \sum_{i=1}^{n} g_i(x)(x_i - x_i^0)$ and, moreover, $g_i(x^0) = \frac{\partial f}{\partial x_i}(x^0)$.*

PROOF. Put

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(x^0 + t(x - x^0))dt.$$

By applying the elementary transformations known from analysis, we obtain

$$f(x) - f(x^0) = \int_0^1 \frac{df(x^0 + t(x - x^0))}{dt} dt = \int_0^1 \left(\sum_{i=1}^{n} (x_i - x_i^0) \frac{\partial f}{\partial x_i}(x^0 + t(x - x^0)\right) dt$$

$$= \sum_{i=1}^{n} (x_i - x_i^0) \int_0^1 \frac{\partial f}{\partial x_i}(x^0 + t(x - x^0))dt = \sum_{i=1}^{n} (x_i - x_i^0) g_i(x). \ ■$$

Exercise 4°. Let f be a function of class C^{r+2}, $r \geqslant 0$, given on a convex neighbourhood $V(x^0)$ of a point x^0 in R^n. Show that

$$f(x) = f(x^0) + \sum_{i=1}^{n} (x_i - x_i^0)\frac{\partial f}{\partial x_i}(x^0) + \sum_{i,j=1}^{n} (x_i - x_i^0)(x_j - x_j^0)A_{ij}(x),$$

where $A_{ij}(x)$ are functions of class C^r on $V(x^0)$, and, in addition, $A_{ij}(x^0) = \dfrac{1}{2}\dfrac{\partial^2 f}{\partial x_i \partial x_j}(x^0)$.

2. SMOOTH SUBMANIFOLDS IN EUCLIDEAN SPACE

1. The Notion of Smooth Submanifold in R^n. In the courses of analysis and analytical geometry, smooth surfaces in the three-dimensional Euclidean space that are given by an equation $z = f(x, y)$, where f is a smooth function of two variables defined in a region D of the plane (x, y), are considered. More complex surfaces (e.g., closed) which are given on their particular regions (i.e., locally) by one of the following equations $z = p(x, y)$, $y = q(x, z)$, $x = r(y, z)$ are also considered. The simplest example of such a surface is the sphere S^2. Other objects studied in analysis and analytical geometry are smooth curves given locally by one of the systems of equations:

$$\begin{cases} x = \varphi(z), \\ y = \psi(z); \end{cases} \quad \begin{cases} x = \varphi(y), \\ z = \psi(y); \end{cases} \quad \begin{cases} y = \varphi(x), \\ z = \psi(x). \end{cases}$$

All these objects are embraced by the single notion of smooth submanifold in a Euclidean space.

We now consider a subset M in R^N as a topological space equipped with the topology induced from R^N. Let x be a point of M, and $U(x)$ its open neighbourhood (in M).

DEFINITION 1. If a homeomorphism $\varphi : R^n \to U(x)$, $n \leqslant N$, satisfying the conditions (i) $\varphi \in C^r$, $r \geqslant 1$, as a mapping from R^n to R^N, (ii) $\operatorname{rank}_y \varphi = n$ for any point $y \in R^n$ is given, then the pair $(U(x), \varphi)$ is called a *chart at the point x on M of class C^r*, or a *C^r-chart on M.*

NOTE. It follows from definition that the C^r-chart $(U(x), \varphi)$ at a point x is a C^r-chart at any point $y \in U(x)$. This explains the fact that the pair $(U(x), \varphi)$ is also called a C^r-chart on M.

Thus, to specify a chart means to specify locally the set M (i.e., to specify the neighbourhood $U(x)$) in the form

$$\begin{aligned} x_1 &= \varphi_1(y_1, \ldots, y_n), \\ x_2 &= \varphi_2(y_1, \ldots, y_n), \\ &\cdots\cdots\cdots\cdots \\ x_N &= \varphi_N(y_1, \ldots, y_n), \end{aligned} \tag{1}$$

where φ_i, $i = 1, \ldots, N$ are the functions of class C^r, which determine the homeomorphism φ.

The neighbourhood $U(x)$ is often called a *coordinate neighbourhood* in view of the fact that homeomorphism (1) determines the curvilinear coordinates $y_1, \ldots, y_n$ on the set $U(x)$, which are not, generally speaking, related to the standard coordinates of the ambient space R^N.

DEFINITION 2. A mapping $f : A \to R^n$ of a subset $A \subset R^N$ into the space R^n is called a *C^r-mapping*, $r \geqslant 1$, on A ($f \in C^r(A)$) if for each point $x \in A$, there exist an open neighbourhood $\tilde{U}(x)$ in the space R^N and a C^r-mapping $\tilde{f} : \tilde{U}(x) \to R^n$ such that $\tilde{f}|_{A \cap \tilde{U}(x)} = f$.

LEMMA 1. *The homeomorphism φ^{-1} from Definition 1 is of class C^r.*

PROOF. Let $x \in U(x_0) \subset R^N$. Then $\varphi^{-1}(x) \in R^n$. Since rank $_{\varphi^{-1}(x)}\varphi = n$, by the theorem on rectifying a mapping (see Sec. 1), there exist an open neighbourhood $W(x) \subset R^N$ of the point x, an open set $W_1 \subset R^N$ and a C^r-diffeomorphism $F : W(x) \to W_1$ such that $F\varphi$ on a certain neighbourhood $V(\varphi^{-1}(x)) \subset R^n$ of the point $\varphi^{-1}(x)$ is the standard embedding of R^n into R^N. Let $g : R^N \to R^n$ be the standard projection, it being evident that $g \in C^r$. Considering the mapping $gF : W(x) \to R^n$, we obtain that $gF \in C^r$ and $gF|_{W(x) \cap U(x_0)} = \varphi^{-1}$. ■

Definition 2 enables us to generalize the notion of diffeomorphism of open sets of the space R^n (see Item 3, Sec. 1) as follows: a homeomorphism $f : A \to B$ of subsets $A \subset R^n$, $B \subset R^m$ is called a *C^r-diffeomorphism* if $f \in C^r(A)$, $f^{-1} \in C^r(B)$.

It follows from Lemma 1 that conditions (i) and (ii) of Definition 1 of a C^r-chart $(U(x), \varphi)$ are equivalent to φ being a C^r-diffeomorphism.

Now we give the basic definition.

DEFINITION 3. A set $M \subset R^N$ is called an *n-dimensional submanifold* in R^N of class C^r or a *C^r-submanifold* if each of its points possesses a certain C^r-chart.

We will denote this submanifold by M^n and write $M^n \in C^r$, indicating thus that it is of class C^r. In other words, a set M in R^N is an n-dimensional submanifold if for each of its points, a coordinate system may be constructed; each coordinate system is determined locally (and called a *local coordinate system*), but the whole set of coordinate systems 'embraces' the entire submanifold.

The definition of a chart may be extended to the case when $r = 0$, omitting condition (ii) in Definition 1. It is natural that in this case nothing can be deduced as regards the differentiability of the homeomorphism. They say then that M^n is a *topological manifold* and write $M^n \in C^0$.

Note that for any point x of a submanifold M^n of class C^r, an infinite, generally speaking, number of charts is determined. Such a set of charts $\{(U_\alpha, \varphi_\alpha)\}$ of class C^r whose open sets $\{U_\alpha\}$ form a covering of M^n is called an *atlas for the submanifold* M^n. The atlas $\{(U_\alpha, \varphi_\alpha)\}$ for the manifold M^n determines the set of coordinate systems that 'serve' the whole submanifold. To specify a submanifold, it suffices to specify an atlas.

Exercises.

1°. Show that if two charts (U, φ), (V, ψ) such that $U \cap V \neq \varnothing$ on a C^r-submanifold M^n are given, then the mapping $\psi^{-1}\varphi : \varphi^{-1}(U \cap V) \to \psi^{-1}(U \cap V)$ of open sets of the space R^n is a C^r-diffeomorphism.

2°. Show that if a sufficient number of replicas of the space R^n are 'glued together' by means of the homeomorphisms $\psi^{-1}\varphi$ that are determined by the charts of a certain atlas, then we obtain a topological space homeomorphic to M^n. (Cf. the development of a two-dimensional surface, see Sec.4, Ch. II).

Thus, the choice of an atlas determines a 'sewing' of a submanifold M^n from n-dimensional spaces by means of a system of charts.

2. Examples of Submanifolds.

(1) A pair $(R^1, 1_{R^1})$, where $1_{R^1} : R^1 \to R^1$ is the identity mapping, determines one C^∞-chart for all $x \in R^1$ and makes the atlas for a one-dimensional submanifold in R^1 of class C^∞.

Exercise 3° Show that a pair (R^1, φ), where $\varphi : R^1 \to R^1$, is determined by the formula $\varphi(x) = x^3$, is not a C^r-chart $(r \geqslant 1)$ at the point $x = 0$, but makes the atlas for a one-dimensional manifold in R^1 of class C^0.

(2) Similarly to Example 1, a pair $(R^n, 1_{R^n})$, where $1_{R^n} : R^n \to R^n$ is the identity mapping, determines one C^∞-chart for all $x \in R^n$ and makes the atlas for an n-dimensional submanifold in R^n of class C^∞.

(3) Using the stereographic projection (Fig. 75), we specify an atlas consisting of two charts on the sphere $S^2 \subset R^3$. Then the sets $U_1 = S^2 \setminus \{N\}$, $U_2 = S^2 \setminus \{S\}$ form an open covering of the sphere. The stereographic projections from the North and South poles are of the following forms:

$$\varphi_1(x) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}\right),$$

$$\varphi_2(x) = \left(\frac{x_1}{1 + x_3}, \frac{x_2}{1 + x_3}\right),$$

respectively, and are homeomorphisms from U_1, U_2 onto R^2.

Exercise 4°. Verify that the mappings φ_1^{-1}, φ_2^{-1} are of class C^∞, and that in each point $y \in R^2$, $\operatorname{rank}_y \varphi_1^{-1} = \operatorname{rank}_y \varphi_2^{-1} = 2$.

Thus, the sphere S^2 with the atlas consisting of two charts (U_1, φ_1^{-1}), (U_2, φ_2^{-1}) is a two-dimensional submanifold in R^3 of class C^∞.

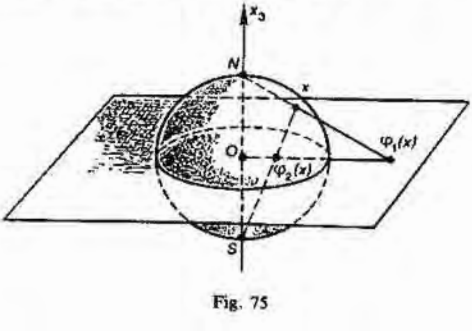

Fig. 75

(4) By means of the stereographic projection (see Example 3), an atlas consisting of two charts (U_1, φ_1^{-1}), (U_2, φ_2^{-1}), where

$$\varphi_1(x) = \left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}}\right),$$

$$\varphi_2(x) = \left(\frac{x_1}{1 + x_{n+1}}, \ldots, \frac{x_n}{1 + x_{n+1}}\right),$$

can be given on the sphere S^n. The sphere S^n with such an atlas is an n-dimensional submanifold in R^{n+1} of class C^∞.

(5) The graph of a mapping. Let a mapping $f : R^n \to R^m, f \in C^r$ be given. Consider the graph $\Gamma(f) = \{x, f(x)\} \subset R^n \times R^m$ of the mapping (see Ex. 12, Sec. 9, Ch. II). The atlas on $\Gamma(f)$ will be given to consist of one chart (R^n, φ), where $\varphi : R^n \to R^{n+m}$ is defined by the formula $\varphi(x) = (x, f(x))$. Thus $\Gamma(f)$ is an n-dimensional submanifold in R^{n+m} of class C^r.

Exercises.

5°. Show that the set of points $\left(x, \sin\frac{1}{x}\right)$ of the plane $R^2, x \in R^1, x \neq 0$, is a one-dimensional submanifold in R^2 of class C^∞.

6°. Show that any set in R^n, consisting of isolated points, is the zero-dimensional submanifold in R^n.

(6) The solution set of a system of equations. Let there be given a system of equations

$$f_1(x_1, \ldots, x_n) = 0,$$
$$f_m(x_1, \ldots, x_n) = 0,$$

where $f_1, \ldots, f_m : R^n \to R^1$ are functions of class C^r, $r \geqslant 1$. Let $n \geqslant m$. The family of functions $f_1, \ldots, f_m$ determines a C^r-mapping $f : R^n \to R^m$. The solution set of the system will be denoted by M. It is clear that $M = f^{-1}(0)$.

THEOREM 1. *Let a set M be nonempty. If for each point $x \in M$, the rank of the Jacobian matrix $\left.\left(\frac{\partial f}{\partial x}\right)\right|_x$ equals m, then M is an $(n - m)$-dimensional submanifold in R^n of class C^r.*

PROOF. Let x_0 be an arbitrary point of M. From the data of the theorem, x_0 is a regular point of the mapping f. According to the theorem on rectifying a mapping (see Sec. 1), there exist an open neighbourhood $V(x_0) \subset R^n$ of the point x_0, an open set $W \subset R^m$ and a C^r-diffeomorphism $F : V(x_0) \to W$ such that fF^{-1} on the set W is the standard projection of R^n onto R^m. Without loss of generality, we may assume that W is a certain open disc $D^n_\rho(y_0)$, $y_0 = F(x_0)$; then

$$(fF^{-1})^{-1}(0) \cap D^n_\rho(y_0) = R^{n-m} \cap D^n_\rho(y_0) = D^{n-m}_\rho(y_0).$$

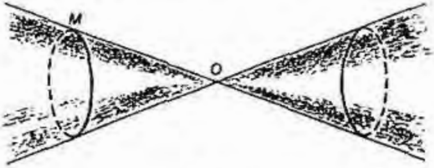

Fig 76

(Here $R^{n-m} = \{x \in R^n : x_1 = \ldots = x_m = 0\}$ is a subspace in R^n). It is clear that a disc $D_\rho^{n-m}(y_0)$ which is open in R^{n-m} is the image of the set $M \cap V(x_0)$ under the diffeomorphism F. Thus, the open neighbourhood $V(x_0) \cap M$ of the point x_0 in M is C^r-diffeomorphic to the open disc $D_\rho^{n-m}(x_0)$, and therefore to the space R^{n-m}. ■

As an example, consider again the sphere $S^n \subset R^{n+1}$, determining it by the equation $x_1^2 + \ldots + x_{n+1}^2 - 1 = 0$. Here rank $\left(\frac{\partial f}{\partial x}\right)$ at any point of the sphere equals unity. Therefore, the conditions of Theorem 1 are fulfilled (for any $r \geqslant 1$). Thus, we have proved once again that the sphere S^n is an n-dimensional submanifold in R^{n+1} of class C^∞.

Consider the case of the conditions of Theorem 1 not being fulfilled. Let a set $M \subset R^3$ be given by the equation $x_1^2 - x_2^2 - x_3^2 = 0$ (Fig. 76). On the set $M \setminus 0$, the structure of the two-dimensional C^∞-submanifold may be specified as before. Besides, at the point 0, all minors of the Jacobian matrix are zero, and its rank is not maximal. The set M is a simple example of an algebraic manifold, and the point 0 a singular point of this manifold.

3. SMOOTH MANIFOLDS

1. The Notion of Smooth Manifold. This notion is one of the central notions of smooth topology and modern analysis. The method of introduction of coordinates on a set may be generalized without assuming that it lies in the space R^N. The development of this idea leads to the notion of smooth manifold.

Let M be a topological space, $U \subset M$ an open set, and $\varphi : R^n \to U$ a homeomorphism. Then the standard coordinates $\{\xi_1(x), \ldots, \xi_n(x)\}$ of the point $\varphi^{-1}(x)$ in the space R^n are naturally assumed to be the coordinates of a point $x \in U$. Thus, the homeomorphism φ determines the coordinates on a part U of the space M. The pair (U, φ) is called a *chart* on M. For any point $x \in U$, the chart (U, φ) will also be termed the *chart at the point x*.

Let $(U, \varphi)(\varphi : R^n \to U)$, $(V, \psi)(\psi : R^n \to V)$ be two charts on M, and $U \cap V \neq \varnothing$. Then to each point $x \in U \cap V$, there correspond two systems of coordinates

$$\{\xi_1(x), \ldots, \xi_n(x)\} \text{ and } \{\eta_1(x), \ldots, \eta_n(x)\},$$

the coordinates of the points $\varphi^{-1}(x) \in \varphi^{-1}(U \cap V)$ and $\psi^{-1}(x) \in \psi^{-1}(U \cap V)$ which, generally speaking, are different. Both systems of coordinates are 'equal in rights' in the sense that there exists a transition homeomorphism

$$\psi^{-1}\varphi : \varphi^{-1}(U \cap V) \to \psi^{-1}(U \cap V)$$

relating both systems of coordinates and permitting to express the former coordinates continuously in terms of the latter:

$$\begin{aligned} \xi_1 &= \chi(\eta_1, \ldots, \eta_n) \\ \xi_n &= \chi_n(\eta_1, \ldots, \eta_n) \end{aligned} \tag{1}$$

and, conversely, continuously express the latter in terms of the former:

$$\begin{aligned} \eta_1 &= \varkappa_1(\xi_1, \ldots, \xi_n), \\ &\ldots\ldots\ldots\ldots\ldots \\ \eta_n &= \varkappa_n(\xi_1, \ldots, \xi_n). \end{aligned} \tag{2}$$

In formulae (1) and (2), $\chi_1, \ldots, \chi_n, \varkappa_1, \ldots, \varkappa_n$ signify the coordinate functions of the mappings $\varphi^{-1}\psi = (\chi_1, \ldots, \chi_n)$, $\psi^{-1}\varphi = (\varkappa_1, \ldots, \varkappa_n)$.

It sometimes happens to be necessary in analytic problems that dependences (1) and (2) should be differentiable r times, $r = 0, 1, \ldots, \infty$. This means that the homeomorphism $\psi^{-1}\varphi$ is a C^r-diffeomorphism. (For convenience, we call a homeomorphism a C^0-diffeomorphism).

DEFINITION 1. The charts (U, φ), (V, ψ) on M are called *C^r-compatible* if one of the following conditions is fulfilled: (i) $U \cap V = \varnothing$, (ii) $U \cap V \neq \varnothing$ and the homeomorphism $\psi^{-1}\varphi : \varphi^{-1}(U \cap V) \to \psi^{-1}(U \cap V)$ is a C^r-diffeomorphism.

DEFINITION 2. A set of charts $\{(U_\alpha, \varphi_\alpha)\}$ on M is called a *C^r-atlas* or an *atlas of class* C^r if any two of its charts are C^r-compatible and $\bigcup_\alpha U_\alpha = M$.

NOTE. All homeomorphisms φ_α in the definition of a C^r-atlas act from the same space R^n.

Thus, specifying an atlas on M, we thereby introduce coordinates in the neighbourhood of each point $x \in M$ that are called local coordinates.

Let a C^r-atlas $\{(U_\alpha, \varphi_\alpha)\}$ be given on M and, therefore, local coordinates be introduced in the neighbourhood of each point. In many problems of analysis for various reasons it is convenient to introduce new local coordinates (by means of a certain chart (V, ψ)) which should be 'equal in rights' with respect to the original coordinates given by the charts of the C^r-atlas $\{(U_\alpha, \varphi_\alpha)\}$. Thus, the chart (V, ψ) should be C^r-compatible with each chart $(U_\alpha, \varphi_\alpha)$ of the given C^r-atlas. If this condition is fulfilled then the set of charts $\{(U_\alpha, \varphi_\alpha)\} \cup (V, \psi)$ is a C^r-atlas. Since a C^r-atlas $\{(U_\alpha, \varphi_\alpha)\} \cup (V, \psi)$ is derived from the C^r-atlas $\{(U_\alpha, \varphi_\alpha)\}$ by adding an 'equal-in-rights' chart, it is natural to consider these atlases as equivalent.

DEFINITION 3. Two C^r-atlases $\{(U_\alpha, \varphi_\alpha)\}$ $\{(V_\beta, \psi_\beta)\}$ are said to be *equivalent* if any two charts $(U_\alpha, \varphi_\alpha)$, (V_β, ψ_β) are C^r-compatible. In other words, two C^r-atlases are equivalent if their union is a C^r-atlas.

Exercise 1°. Show that the relation which we introduced on the set of C^r-atlases is an equivalence relation.

It follows from Exercise 1° that the set of C^r-atlases on M decomposes into disjoint classes of equivalent atlases.

DEFINITION 4. The equivalence class of C^r-atlases on M is called a *C^r-structure* on M.

Each equivalence class of C^r-atlases on M is determined by any of its representatives, i.e., a given C^r-structure can be restored by any of its C^r-atlases. This remark underlies the fact that a C^r-structure on M may be specified by specifying on it one C^r-atlas from the given C^r-structure.

The union of all C^r-atlases from a given C^r-structure is also a C^r-atlas called *maximal*. Specifying a C^r-structure is equivalent to specifying the maximal atlas. Sometimes the maximal atlas is called a C^r-structure.

C^0-structures are termed *topological structures*; C^r-structures ($r = 1, \ldots, \infty$) are called *smooth* (or *differential*) structures.

DEFINITION 5. A topological space M with a C^r-structure given on it is called a *C^r-manifold* (or a *manifold of class C^r*), and the dimension of the space R^n from which the homeomorphisms of charts act is called the *dimension* of the C^r-manifold.

Similarly to C^r-structures, C^0-manifolds are said to be *topological* and C^r-manifolds ($r = 1, \ldots, \infty$) *smooth*. Sometimes (for brevity) C^r-manifolds will be simply referred to as *manifolds*, and C^r-atlases as *atlases*.

If in condition (ii) of Definition 1, the homeomorphisms $\psi^{-1}\varphi$, $\varphi^{-1}\psi$ are analytic mappings ($\psi^{-1}\varphi, \varphi^{-1}\psi \in C^\omega$), then the charts (U, φ), (V, ψ) on M are said to be *C^ω-compatible*. C^ω-atlases, C^ω-structures and C^ω-manifolds are defined naturally. C^ω-structures and C^ω-manifolds are called *analytic structures* and *analytic manifolds*, respectively. To indicate the dimension of a manifold, we will write M^n, and also $\dim M = n$.

NOTE. The dimension of a C^0-manifold is its invariant, i.e., independent of the choice of an atlas. In fact, if M admitted atlases

$$[(U_\alpha, \varphi_\alpha)](\varphi_\alpha : R^n \to U_\alpha),\ [(V_\beta, \psi_\beta)](\psi_\beta : R^m \to V_\beta)$$

and $n \neq m$, then there would be sets U_α, V_β such that $U_\alpha \cap V_\beta \neq \varnothing$ and the mapping

$$\psi_\beta^{-1}\varphi_\alpha : \varphi_\alpha^{-1}(U_\alpha \cap V_\beta) \to \psi_\beta^{-1}(U_\alpha \cap V_\beta)$$

would be a homeomorphism. This is contrary to the Brouwer theorem stating that nonempty open sets $U \subset R^n$, $V \subset R^m$ may be homeomorphic only in case when $n = m$. (This theorem will be proved independently of the subject-matter of this chapter in Sec. 6, Ch. V). For C^r-manifolds, $r \geqslant 1$, the correctness of the definition of a dimension is evident.

Note that a C^0-structure on any space M is unique (this follows from the definition); but if $r \neq 0$ then M may admit several different C^r-structures. In fact, the atlas consisting of one chart (U, φ), where $U = R^1$, and $\varphi : R^1 \to R^1$ is the identity mapping, determines the structure of a C^∞-manifold on R^1. The atlas consisting of one chart (R^1, φ), where $\varphi(x) = x^3$, also determines the structure of a C^∞-manifold

on R^1. It is easy to verify that the atlases considered are not equivalent and, therefore, the C^∞-structures determined by them are different.

It has been proved, moreover, that if there exists on M at least one C^r-structure ($r \geqslant 1$), then there exist infinitely many C^r-structures on M.

Exercises.

2°. Show that the atlases

$$\{(R^1, \varphi_1)\}, \ldots, \{(R^1, \varphi_k)\}, \ldots, \text{ where } \varphi_k(x) = x^{2k+1}, k = 0, 1, \ldots,$$

specify different C^∞-structures on R^1.

3°. Show that any C^r-submanifold in R^N is a C^r-manifold (see Ex. 1, Sec. 2).

We list another formal generalization of the notion of chart (U, φ) when the homeomorphism φ acts from a certain open set of the space R^n, which,generally speaking, does not coincide with the whole space. In this case, all the notions introduced above can be defined by exactly the same formulations. However, this does not lead to generalization of the notion of C^r-manifold. In fact, in this C^r-structure, a C^r-atlas $\{(U_\alpha, \varphi_\alpha)\}$ can be picked such that all homeomorphisms φ_α act on it from open discs D_α in the space R^n. Since there exists a C^r-diffeomorphism $f_\alpha : R^n \to D_\alpha$, the C^r-atlas $\{(U_\alpha, \varphi_\alpha f_\alpha)\}$ is contained in our C^r-structure and consists of usual charts.

In some cases, it is simpler to specify an atlas consisting of generalized charts. We will use this circumstance if necessary without reserve.

EXAMPLES.

1. Any open set V of a manifold M^n of class C^r is itself a manifold of class C^r supplied with the structure determined by the atlas $\{(U_\alpha \cap V, \varphi_\alpha|_{\varphi_\alpha^{-1}(U_\alpha \cap V)})\}$, where $\{(U_\alpha, \varphi_\alpha)\}$ is a certain atlas from the C^r-structure given on M^n.
2. Specify on $S^2 \subset R^3$ a C^∞-atlas consisting of six charts. Put

$$U_k^+ = \{x = (x_1, x_2, x_3) \in S^2 : x_k > 0\}, \ U_k^- = \{x \in S^2 : x_k < 0\}, k = 1, 2, 3.$$

Specify the homeomorphisms $\varphi_k^+ : R^2 \to U_k^+, \varphi_k^- : R^2 \to U_k^-$:

$$\begin{aligned} \varphi_1^+, \varphi_1^- &: (x_2, x_3) \to (\pm \sqrt{1 - x_2^2 - x_3^2}, x_2, x_3), \\ \varphi_2^+, \varphi_2^- &: (x_1, x_3) \to (x_1, \pm \sqrt{1 - x_1^2 - x_3^2}, x_3), \\ \varphi_3^+, \varphi_3^- &: (x_1, x_2) \to (x_1, x_2, \pm \sqrt{1 - x_1^2 - x_2^2}), \end{aligned}$$

where the sign on the right-hand side is chosen in accordance with the '+' or '−' sign on the left-hand side.

Similarly, a C^∞-atlas consisting of $2(n + 1)$ charts can be specified on the sphere S^n. ♦

To make a great number of constructions in the study of topological spaces possible,we need the properties to be Hausdorff and to be countable (for a base for a topology). These properties do not, generally speaking, follow from the definition of a manifold. This can be illustrated by the following examples.

EXAMPLES. 3. A non-Hausdorff manifold M^1 of class C^∞. Consider the interval (0, 3) and break it into three sets (0, 1], (2, 3), (1, 2]. We introduce a topology on their

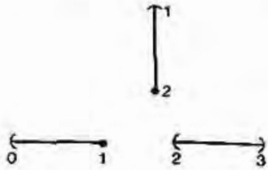

Fig. 77

formal (disjoint) union (Fig. 77) as follows: the neighbourhoods of points in the set $(0, 1) \cup (1, 2) \cup (2, 3)$ are the same as in the topology induced by the real straight line. As to the points $x_1 = 1$, $x_2 = 2$, their neighbourhoods are the sets $(1 - \varepsilon, 1] \cup (2, 2 + \varepsilon)$, $(2 - \varepsilon, 2] \cup (2, 2 + \varepsilon)$. Then the points x_1, x_2 are not separated. We leave it as an exercise to show that the structure of a one-dimensional C^∞-manifold may be specified on the obtained space in a natural fashion and that this manifold possesses a countable base.

4. A manifold M^1 of class C^∞ not possessing a countable base. Considering the set $M = R^1 \times R^1$, we define a topology on M as the Cartesian product topology, where the first factor R^1 is endowed with the usual topology, and the second with the discrete. It is easy to show that this is a Hausdorff one-dimensional manifold of class C^∞, whose topology does not possess a countable base.

On the basis of the last two examples, a non-Hausdorff manifold without a countable base can be constructed easily (by taking their Cartesian product).

Example 4 contradicts the intuitive idea of the plane being a manifold of dimension 2. This circumstance, even taken into account independently, impels us to impose on a manifold the condition that the base should be countable.

The manifold M^n is usually assumed to be Hausdorff and satisfying the second countability axiom. We will also do it without further notice. Then, it is easy to prove that the manifold M^n is a locally compact and even paracompact space.

In fact, the local compactness follows from the following simple exercise.

Exercise 4°. Show that if (U, φ) is a chart on M^n, $x \in U$ and $D^n(\varphi^{-1}(x))$, $\bar{D}^n(\varphi^{-1}(x))$ are an open and a closed disc, respectively, in R^n with the centre at the point $\varphi^{-1}(x)$ and radius 1, then $\varphi(D^n(\varphi^{-1}(x)))$ is a neighbourhood of the point x, open in M^n and whose closure (in M^n) is compact and equals $\varphi(\bar{D}^n(\varphi^{-1}(x)))$.

The paracompactness of the manifold M^n follows from its local compactness and the base countability (due to the corollary to Theorem 5, Sec. 13, Ch. II).

2. Projective Spaces. The definition and different topologically equivalent interpretations of the projective spaces RP^{n-1}, CP^{n-1}, $n \geq 2$, were given in Item 2, Sec. 5, Ch. II (see also Item 1, Sec. 3, Ch. I). The structures of C^∞-manifolds may be defined on the spaces RP^{n-1} and CP^{n-1}. We illustrate the idea of introducing the local coordinates on RP^{n-1} below. Consider RP^{n-1} as the set $L = \{l\}$ of all straight lines of the space R^n passing through the origin. Each straight line intersects one or more hyperplanes of the form $x_j = 1$. Fix one of these hyperplanes $x_i = 1$ and pick out of L the collection U_i of all straight lines intersect-

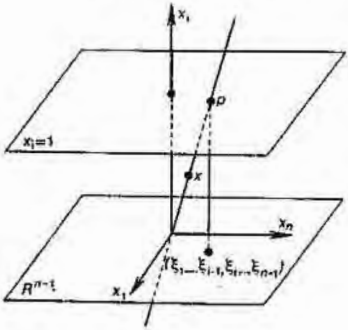

Fig. 78

ing the hyperplane $x_i = 1$. Then the position of the straight line $l \in U_i$ is determined by the Cartesian coordinates $(\xi_1, \ldots, \xi_{i-1}, 1, \xi_i, \ldots, \xi_{n-1})$ of its point of intersection with the hyperplane $x_i = 1$. Naturally, the coordinates $(\xi_1, \ldots, \xi_{i-1}, \xi_i, \ldots, \xi_{n-1})$ may be taken as the local coordinates of the straight line l (see Fig. 78). Thus, we have the homeomorphisms

$$\psi_i(l) = (\xi_1, \ldots, \xi_{i-1}, \xi_i, \ldots, \xi_{n-1}) : U_i \to R^{n-1}, i = 1, \ldots, n.$$

The local coordinates $\xi_1, \ldots, \xi_{n-1}$ are also called the projective coordinates of the straight line l. It is not difficult to express the local coordinates of the straight line l in terms of the coordinates of an arbitrary point $x = (x_1, \ldots, x_n)$ of the straight line l: $\xi_1 = x_1/x_i, \ldots, \xi_{i-1} = x_{i-1}/x_i, \xi_i = x_{i+1}/x_i, \ldots, \xi_{n-1} = x_n/x_i$.

The atlas of n charts (U_i, φ_i), $i = 1, \ldots, n$, where $\varphi_i = \psi_i^{-1}$, determines the structure of a C^∞-manifold of dimension $n - 1$ on RP^{n-1}. We now show the C^∞-compatibility of the charts of the atlas constructed. In fact, let $l \in U_i \cap U_j$ and $\eta_1 = x_1/x_j, \ldots, \eta_{j-1} = x_{j-1}/x_j, \eta_j = x_{j+1}/x_j, \ldots, \eta_{n-1} = x_n/x_j$ be the local coordinates of the straight line l in the chart (U_j, φ_j). For definiteness, let $i < j$. Then the following relations are evident

$$\eta_1/\eta_i = \xi_1, \ldots, \eta_{i-1}/\eta_i = \xi_{i-1}, \eta_{i+1}/\eta_i = \xi_i, \ldots,$$
$$\eta_{j-1}/\eta_i = \xi_{j-2}, 1/\eta_i = \xi_{j-1}, \eta_j/\eta_i = \xi_j, \ldots, \eta_{n-1}/\eta_i = \xi_{n-1},$$

which obviously show that the local coordinates $\xi_1, \ldots, \xi_{n-1}$ infinitely smoothly depend on the coordinates $\eta_1, \ldots, \eta_{n-1}$.

Exercise 5°. Verify that for the projective space RP^{n-1} considered as the collection of pairs of diametrically opposite points of the sphere S^{n-1}, local coordinates may also be given in the above manner.

Exercise 6°. Show that the complex projective space CP^{n-1} possesses a C^∞-atlas converting it into a C^∞-manifold of real dimension $2n - 2$.

Hint: Considering CP^{n-1} as the set of complex straight lines in C^n, specify the atlas by means of the formulae similar to those in the case of RP^{n-1}.

A certain manifold M^n of class C^r on which the group Z_k acts can be considered in a more generalized manner (see Sec. 5, Ch. II). We will assume that the orbit of each point consists, under this action, of k different elements.

Exercise 7°. Let $h : Z_k \to H(M^n)$ be a homomorphism of the group Z_k (k being prime) into the group of homeomorphisms of M^n, determining the action of Z_k in M^n, and let g be the generator of the group Z_k. Show that the condition $h_g(x) \neq x$ for any $x \in M^n$ is equivalent to the assumption that the orbit of every point consists, under this action, of k different elements. In this case, the group Z_k is said to *act without fixed points*.

Assume, further, that the charts of the form $(h_g U_\alpha, h_g \varphi_\alpha)$ are C^r-compatible with the charts (U_β, φ_β) of the C^r-atlas on M^n.

Consider the factor space M^n/Z_k. It is also a C^r-manifold of dimension n. The atlas is specified as follows: let O_x be the orbit of the point x, and $U(O_x)$ the neighbourhood of the orbit in M^n/Z_k, consisting of all the orbits O_y passing through the points y of a sufficiently small neighbourhood $V(x)$ of a point x in M^n ($V(x)$ must not contain any pairs of points y, $h_g(y)$ and must lie wholly in some chart of the manifold M^n). Then the local coordinates of the point $y \in V(x)$ in $V(x)$ will be called the local coordinates of the orbit $O_y \in U(O_y)$. This may be seen to be a C^r-atlas.

Note that the condition for the charts $(h_g U_\alpha, h_g \varphi_\alpha)$ and (U_β, φ_β) to be compatible is not burdensome (see below Theorem 2, Sec. 5).

Exercise 8°. Verify that the lens space $L(k, k_1, \ldots, k_n)$ is a C^∞-manifold of dimension $2n + 1$.

3. **Induced Structures.** Let M^n be a C^r-manifold, and $f : M^n \to N$ a homeomorphism of topological spaces M^n and N. The structure of a C^r-manifold called the *structure induced by* f may be introduced on N in a natural manner. Viz., if $\{(U_\alpha, \varphi_\alpha)\}$ be a C^r-atlas for the manifold M^n, then $\{(f(U_\alpha), f\varphi_\alpha)\}$ is a C^r-atlas on N.

Exercise 9°. Verify that $\{(f(U_\alpha), f\varphi_\alpha)\}$ is, in fact, the atlas determining on N the structure of a C^r-manifold of dimension n.

The described method of specifying a structure happens to be quite useful while specifying the structure of a C^r-manifold on the topological space N: we can specify the structure of a C^r-manifold on a 'simpler' space M homeomorphic to N, and then induce the structure of a C^r-manifold on N. The C^∞-structure is given in this manner, e.g., to various models of RP^{n-1}.

EXAMPLES. 5. It is easy to see that any one-dimensional compact C^0-manifold is triangulable. Then any connected, one-dimensional, compact C^0-manifold is homeomorphic to the circumference S^1 (see Ex. 6°, Sec. 4, Ch. II) and therefore the C^∞-structure is naturally induced on it.

6. A two-dimensional, orientable, closed surface, as shown in Sec. 4, Ch. II, is homeomorphic to an M_p-surface (i.e., a sphere with p handles) which can be realized in the space R^3 as a C^∞-submanifold (which is intuitively obvious). Thus, orientable closed surfaces are endowed with the C^∞-manifold structure.

Exercises.

10°. Specify the C^∞-manifold structure on the boundary of the cube $I^n = \{x = (x_1, \dots, x_n) : |x_i| \leqslant 1, i = 1, \dots, n\}$, inducing it from the sphere S^{n-1}.

11°. Show that the mapping

$$(x_1, x_2, x_3) \to (x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3)$$

is a homeomorphism of the projective plane RP^2 onto a subset in R^6. To induce the structure of the smooth manifold RP^2 by this homeomorphism means to realize thereby RP^2 as a subset in R^6.

12°. Construct the realization of RP^3 in R^{10}.

4. Matrix Manifolds. We endow the set $M(m, n)$ of all $m \times n$ matrices with elements from R^1 with the topology induced by the natural mapping $i : R^{mn} \to M(m, n)$:

$$(x_1, \dots, x_{mn}) \to \begin{pmatrix} x_1 & \dots x_n \\ x_{n+1} & \dots x_{2n} \\ \dots & \dots \\ x_{(m-1)n+1} & \dots x_{mn} \end{pmatrix}.$$

Then the homeomorphism i induces on $M(m, n)$ the structure of a C^∞-manifold of dimension mn.

Denoting the subspace of matrices of a fixed rank k in $M(m, n)$ by $M(m, n; k)$, we specify the structure of a C^∞-manifold of dimension $k(m + n - k)$ on $M(m, n; k)$. Note, beforehand, that if $Y \in M(m, n)$ and rank $Y \geqslant k$, then by interchanging the rows and columns, the matrix Y may be transformed to the form

$$\left(\begin{array}{c|c} A_Y & B_Y \\ \hline C_Y & D_Y \end{array}\right),$$

where A_Y is a non-singular square matrix of order k. In other words, there exist nonsingular square matrices $P_Y \in M(m, m)$, $Q_Y \in M(n, n)$ such that

$$P_Y Y Q_Y = \left(\begin{array}{c|c} A_Y & B_Y \\ \hline C_Y & D_Y \end{array}\right).$$

We show that rank $Y = k$ if and only if $D_Y = C_Y A_Y^{-1} B_Y$. In fact, it follows from the equality

$$\left(\begin{array}{c|c} I_k & 0 \\ \hline -C_Y A_Y^{-1} & I_{m-k} \end{array}\right) \left(\begin{array}{c|c} A_Y & B_Y \\ \hline C_Y & D_Y \end{array}\right) = \left(\begin{array}{c|c} A_Y & B_Y \\ \hline 0 & -C_Y A_Y^{-1} B_Y + D_Y \end{array}\right)$$

that

$$\operatorname{rank} Y = \operatorname{rank} \left(\begin{array}{c|c} A_Y & B_Y \\ \hline 0 & -C_Y A_Y^{-1} B_Y + D_Y \end{array}\right)$$

It can be seen from the latter equality that rank $Y = k$ if and only if $D_Y = C_Y A_Y^{-1} B_Y$.

Now let $X_0 \in M(m, n; k)$ and X an arbitrary matrix from $M(m, n; k)$. Denote

$$P_{X_0} X Q_{X_0} = \left(\begin{array}{c|c} A_{X, X_0} & B_{X, X_0} \\ \hline C_{X, X_0} & D_{X, X_0} \end{array} \right),$$

where A_{X, X_0} is a square matrix of order k. Consider an open neighbourhood

$$V(X_0) = \{X \in M(m, n) : \det A_{X, X_0} \neq 0\}$$

of the matrix X_0 in $M(m, n)$. Then $U(X_0) = V(X_0) \cap M(m, n; k)$ is an open neighbourhood of X_0 in $M(m, n; k)$ and the mapping

$$\varphi_{X_0} : U(X_0) \to R^{mn - (m - k)(n - k)}$$

specified as follows

$$X \to \left(\begin{array}{c|c} A_{X, X_0} & B_{X, X_0} \\ \hline C_{X, X_0} & D_{X, X_0} \end{array} \right) \to \left(\begin{array}{c|c} A_{X, X_0} & B_{X, X_0} \\ \hline C_{X, X_0} & 0 \end{array} \right) \xrightarrow{i} R^{mn - (m - k)(n - k)}$$

is a homeomorphism (i being the natural mapping). Therefore $(U(X_0), \varphi_{X_0}^{-1})$ is a chart. Specifying thus a chart for every matrix $X_0 \in M(m, n; k)$, we obtain a C^∞-atlas on $M(m, n; k)$.

Exercise 13°. Show the C^∞-compatibility of the constructed atlas.

Note that $M(k, n; k)$ may be interpreted as the set of ordered sets of k linearly independent vectors in R^n; $M(n, n; n)$ is denoted by $GL(n; R)$ (general linear group).

5. **Grassmann Manifolds.** One natural generalization of the projective space RP^{n-1} is the Grassmann manifold $G_k(R^n)$ consisting of all k-dimensional subspaces, $k \geqslant 1$, of the space R^n (when $k = 1$, this becomes the projective space). We equip the space $G_k(R^n)$ with the topology induced by the natural mapping $M(k, n; k) \to G_k(R^n)$, associating each matrix

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots \\ x_{k1} & \cdots & x_{kn} \end{pmatrix}$$

with the subspace in R^n spanned by the vectors

$$x_i = (x_{i1}, \ldots, x_{in}),\ i = 1, \ldots, k.$$

Note that $G_k(R^n)$ is homeomorphic to the orbit space of $M(k, n; k)$ with respect to the (left) action of the group $GL(k, R)$; the element $C \in GL(k, R)$ acts on the element $Y \in M(k, n; k)$ in accordance with the rule $Y \to CY$ (the matrix product). In other words, the space $G_k(R^n)$ is homeomorphic to the factor space $M(k, n; k)/R$ with the following equivalence relation: $X \overset{R}{\sim} Y$ if there exists a square non-singular matrix C of order k such that $X = CY$. To specify on $G_k(R^n)$ the structure of a C^∞-manifold, we will equip $M(k, n; k)/R$ with the C^∞-structure and induce the C^∞-structure on $G_k(R^n)$ by the homeomorphism $f : M(k, n; k)/R \to G_k(R^n)$. The manifold that we are to obtain is called the *Grassmann manifold*.

Local coordinates on $M(k, n; k)/R$ may be given by the analogy with the projective space RP^{m-1} if $M(k, n; k)$ is considered in place of $R^m \setminus 0$, the subspace $L = TX$, $T \in GL(k, R)$, $X \in M(k, n; k)$, in place of the straight lines $l = tx$, $t \in R^1 \setminus 0$, $x \in R^m \setminus 0$, and the set $H_{i_1, \ldots, i_k}$ of matrices from $M(k, n, k)$ for which the submatrix made up of columns $i_1, \ldots, i_k$, is a unit matrix is considered in place of the hyperplane $x_i = 1$. The subspace TX intersects the set $H_{i_1, \ldots, i_k}$ if and only if the submatrix $X_{i_1, \ldots, i_k}$ made up of columns $i_1, \ldots, i_k$ of the matrix X is non-singular (i.e., $\det X_{i_1, \ldots, i_k} \neq 0$); in case $X_{i_1, \ldots, i_k}$ is non-singular, the 'intersection point' is, as can be easily seen, the matrix $Y = X^{-1}_{i_1, \ldots, i_k}X$. To determine the charts, fix the set $H_{i_1, \ldots, i_k}$ (i.e., fix the numbers of the columns in the matrix $X \in M(k, n; k)$) and consider the set $U_{i_1, \ldots, i_k}$ of all subspaces TX whose intersection with $H_{i_1, \ldots, i_k}$ is nonempty. In other words, $U_{i_1, \ldots, i_k}$ is the set of subspaces TX such that the submatrix $X_{i_1, \ldots, i_k}$ for the generator X of the subspace TX is non-singular. It is natural to assume the elements of the matrix $Y_{j_1, \ldots, j_{n-k}}$ formed by the columns $j_1, \ldots, j_{n-k}$ of the matrix Y, which are different from $i_1, \ldots, i_k$ to be local coordinates. More precisely, the pair $(U_{i_1, \ldots, i_k}\varphi_{i_1, \ldots, i_k}$, where $\varphi_{i_1, \ldots, i_k}: M(k, n; k)/R \to R^{k(n-k)}$ is a homeomorphism given by the correspondence

$$X \to Y_{j_1, \ldots, j_{n-k}} = \begin{pmatrix} y_{11} & \cdots & y_{1(n-k)} \\ y_{k1} & \cdots & y_{k(n-k)} \end{pmatrix} \xrightarrow{f}$$

$$(y_{11}, \ldots, y_{1(n-k)}, y_{21}, \ldots, y_{k(n-k)})$$

is a chart on $M(k, n; k)/R$.

Exercise 15°. Verify that the mappings $\varphi_{i_1, \ldots, i_k}$ are homeomorphisms.

The atlas $[(U_{i_1, \ldots, i_k}, \varphi^{-1}_{i_1, \ldots, i_k})]$ of C^k_n charts determines the structure of the C^∞-manifold of dimension $k(n-k)$ on $M(k, n; k)/R$.

Exercise 16°. Show that the manifold $G_k(R^n)$ is homeomorphic to the manifold $G_{n-k}(R^n)$.

6. **Products of Manifolds.** If M^n, N^m are C^r-manifolds then the structure of a C^r-manifold of dimension $m + n$ can be naturally specified on the topological product $M \times N$. We leave it to the reader to verify.

Two examples of the products of manifolds are the cylinder $R^1 \times S^1$ and the k-dimensional torus $T^k = S^1 \times \ldots \times S^1$ (k factors). According to the abovesaid, they are C^∞-manifolds of dimensions 2 and k, respectively.

7. **Riemann Surfaces.** Consider an example which is important for the theory of functions of a complex variable. Let M^2 be a two-dimensional smooth manifold. Regard R^2 as the complex z-plane. Let $[(U_\alpha, \varphi_\alpha)]$ be an atlas on M^2 such that the transition diffeomorphisms

$$\varphi_\beta^{-1}\varphi_\alpha : \varphi_\alpha^{-1}(U_\alpha \cap U_\beta) \to \varphi_\beta^{-1}(U_\alpha \cap U_\beta)$$

are complex-valued analytic functions of z in the regions $\varphi_\alpha^{-1}(U_\alpha \cup U_\beta)$. The manifold M^2 with such an atlas is called a *Riemann surface* (abstract). Its complex

analytic structure is determined by an atlas equivalence such that the transition diffeomorphisms are complex-valued analytic functions.

In particular, the complex z-plane C is a Riemann surface, its complex analytic structure being given by the atlas consisting of a unique chart $(C, 1_C)$, where 1_C is the identity mapping.

The sphere $S^2 = \{x \in R^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ is also a Riemann surface. Specify on S^2 an analytic structure $U_1 = S^2 \setminus \{N\}$, $U_2 = S^2 \setminus \{S\}$, the local coordinates of the point $P(x_1, x_2, x_3)$ in U_1, U_2 being, respectively, of the form

$$z_1 = \frac{x_1 + ix_2}{1 - x_3}, \quad z_2 = \frac{x_1 - ix_2}{1 + x_3}.$$

The coordinate z_1 is given rise if the sphere S^2 (see Fig. 75) is stereographically projected on the equatorial plane from the pole N, while z_2 arises if S^2 is projected from the pole S. If $P \in U_1 \cap U_2$ then $z_1 \neq 0$, $z_2 \neq 0$ and, evidently, $z_1 z_2 = 1$. Hence, the transition diffeomorphism $z_1 = 1/z_2$ is an analytic function. The extended z-plane (z-sphere) $\bar{C}$ is endowed with a complex analytic structure by means of the homeomorphism onto S^2.

The two-sheeted Riemann surface of the function $w = \sqrt{z}$ (see Sec. 4, Ch. 1) is a complex analytic manifold, and the analytic structure on it is introduced via the homeomorphism to the z-sphere.

Exercise 17°. Describe the corresponding atlas of the two-sheeted Riemann surface of the function $w = \sqrt{z}$.

It is proved in the theory of functions of a complex variable that any analytic function on the z-plane possesses an abstract Riemann surface and that any compact abstract Riemann surface can be realized as the Riemann surface of a certain algebraic function.

8. The Configuration Space. The considered examples of smooth manifolds emerge naturally in various mathematical problems. The notion of manifold is as naturally used in applied sciences too (e.g., mechanics or physics) to describe the set of positions (i.e., the configuration space) of a system. We adduce the simplest example.

Consider a hinged pendulum swinging in the vertical plane. Denote the point where the pendulum is attached by O, the hinge by O_1, and the end of the pendulum by O_2. Each position of the system is given by the direction of the rod OO_1 and of the rod O_1O_2 or by the pair of angles φ, ψ (Fig. 79) varying independently in the intervals $0 \leqslant \varphi < 2\pi$, $0 \leqslant \psi < 2\pi$. The configuration space of the given system is thus the Cartesian product of two circumferences $S^1 \times S^1$, i.e., the two-dimensional torus T^2.

Exercise 18°. Describe the configuration space of a plane two-hinged pendulum.

More complicated configuration spaces emerge in the study of more complex mechanical systems consisting of a greater number of point masses and undergoing more complex patterns of displacement. These conditions are usually given in the form of equations to which the coordinates of all the point masses should satisfy (these equations are called geometric relations). It is the set of geometric relations that determines (under the corresponding conditions) a smooth manifold in the

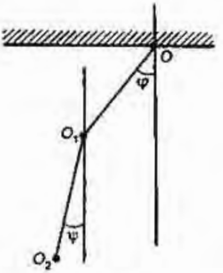

Fig. 79

space R^{3n}, where n is the number of point masses (See Example 6, Sec. 2). An ordered set of coordinates in R^{3n} of n point masses determines a position of a mechanical system in the configuration space.

9. Manifolds with Boundary. The notion of manifold introduced above does not embrace, however, a number of geometric objects, e.g., the n-dimensional closed disc, surfaces with boundary, etc. In fact, it is impossible to find a neighbourhood which is homeomorphic to the space R^n (or its open part) of points on the boundary of the disc $\bar{D}^n$. This drawback is removed by introducing the notion of manifold with boundary.

Consider the subspace R^{n-1} of the space R^n. The former subdivides the space R^n into two half-spaces:

$$R^n_+ = \{x \in R^n : x_n \geqslant 0\} \text{ and } R^n_- = \{x \in R^n : x_n \leqslant 0\},$$

the boundary of each of which being the subspace

$$R^{n-1} = \{x \in R^n : x_n = 0\}.$$

The half-space R^n_+ may serve as the simplest example of an n-dimensional manifold with the boundary R^{n-1}. If now a certain number of half-spaces R^n_+ are 'glued' together taking care to 'glue' a boundary with a boundary, then we will obtain an object called an n-dimensional manifold with boundary, where the boundary is the result of 'gluing' the replicas of the subspaces R^{n-1} together; it is an $(n-1)$-dimensional manifold itself.

Let us describe the manifold with boundary in greater detail. Let M be a topological space, and N its subspace. Extend the notions of chart and atlas. An open set $U \subset M$ and a homeomorphism $\varphi : R^n \to U$ or $R^n_+ \to U$ form a pair (U, φ) called a *chart*. A set of charts $\{(U_\alpha, \varphi_\alpha)\}$ is called a C^r-*atlas* for the pair M, N if (1) $M = \bigcup\limits_\alpha U_\alpha$; (2) if $\varphi_\alpha : R^n_+ \to U_\alpha$, then $\varphi_\alpha|_{R^{n-1}} : R^{n-1} \to N \cap U_\alpha$ (a chart for the subspace N); (3) there is a subset of charts $\{(U_\gamma, \varphi_\gamma)\}$, $\varphi_\gamma : R^n_+ \to U_\gamma$ such that

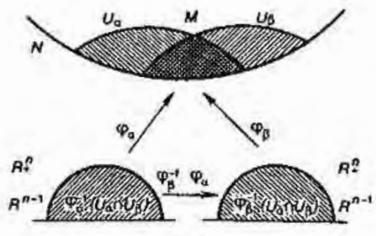

Fig. 80

$N \subset \bigcup\limits_{\gamma} U_\gamma$; (4) the homeomorphisms $\varphi_\beta^{-1}\varphi_\alpha : \varphi_\alpha^{-1}(U_\alpha \cap U_\beta) \to \varphi_\beta^{-1}(U_\alpha \cap U_\beta)$ are diffeomorphisms of class C^r (Fig. 80).

NOTE. In the case when the homeomorphism $\varphi_\beta^{-1}\varphi_\alpha$ acts between sets which are open in R^n_+ but not open in R^n, the properties of being smooth (and diffeomorphic) are understood in the sense of Definition 2, Sec. 2.

It is evident that the set of charts $\{(N \cap U_\gamma), \varphi_\gamma|_{R^{n-1}}\}$ forms a C^r-atlas on N, specifying the structure of a C^r-manifold of dimension $n - 1$ on it. The atlas $\{(U_\alpha, \varphi_\alpha)\}$ is said to determine on M the structure of an n-dimensional *C^r-manifold with a boundary N*. The boundary is usually denoted by ∂M^n.

Exercise 19°. Prove that the sets in R^n given by the inequalities $x_1^2 + \ldots + x_n^2 \leqslant 1, x_1^2 + \ldots + x_n^2 \geqslant 1$ are n-dimensional C^∞-manifolds with the common boundary S^{n-1}.

10. The Existence of Smooth Structures. We now make some remarks regarding the possibility of introducing smooth structures. Whitney has proved that if there exists a C^r-structure ($r \geqslant 1$) on a space M, then there also exists on it a C^∞-structure (and even a C^ω-structure); moreover, a C^∞-atlas may be chosen from the maximal atlas for the given C^r-structure. The exception is the case when $r = 0$. It is known that a C^1-structure may be introduced on any C^0-manifold of dimension $n \leqslant 4$ (and hence a C^∞-structure), but for any $n \geqslant 10$, there exist C^0-manifolds which do not admit the introduction of a C^1-structure.

4. SMOOTH FUNCTIONS IN A MANIFOLD AND SMOOTH PARTITION OF UNITY

This and the subsequent sections are devoted to the construction of the elements of analysis on smooth manifolds.

1. The Notion of Smooth Function in a Manifold. A function defined in a manifold M^n can be considered locally as a function of the local coordinates of a point $x \in M^n$, i.e., as a function of the standard coordinates of the point $\varphi_\alpha^{-1}(x)$ in R^n that are given by a certain chart $(U_\alpha, \varphi_\alpha)$, $x \in U_\alpha$. Thus, we find

ourselves within the range of the notions of analysis, and, in particular, can define and investigate the notion of smooth function.

DEFINITION 1. Let M^n be a manifold of class C^r, $r \geqslant 1$. A mapping $f : M^n \to R^1$ is called a *C^r-function* (a *function of class C^r*) *in a neighbourhood of a point* $x \in M^n$ if there is a chart $(U_\alpha, \varphi_\alpha)$, $(x \in U_\alpha)$ for M^n such that the mapping $f\varphi_\alpha : R^n \to R^1$ is a C^r-mapping onto R^n.

Exercise 1°. Show that the definition of a C^r-function in a neighbourhood of a point does not depend on the choice of a chart.

DEFINITION 2. A function $f : M^n \to R^1$ is called a *C^r-function on a certain set* $A \subset M^n$ if it is a C^r-function in a neighbourhood of each point $x \in A$.

We often have to consider a function given not in the whole manifold M^n, but only on its subset. Definitions 1 and 2 are extended, in a natural manner, to the case of functions $f : U \to R^1$ defined on an open subset $U \subset M^n$ while choosing the charts $(U_\alpha, \varphi_\alpha)$ so that $U_\alpha \subset U$. However, these definitions should be extended if functions $f : A \to R^1$ defined on an arbitrary subset $A \subset M^n$ are considered.

DEFINITION 3. A function $f : A \to R^1 (A \subset M^n)$ is called a *C^r-function on A*, if for any point $y \in A$, there exists an open neighbourhood $U(y) \subset M^n$ of the point y and C^r-function $\varphi_y : U(y) \to R^1$ such that $\varphi_y = |_{U(y) \cap A} = f|_{U(y) \cap A}$.

It is easy to see that each of the local coordinates $\xi_i(x)$, $i = 1, \ldots, n$, of a C^r-manifold is a C^r-function in its domain.

In a special case of two-dimensional manifolds, viz., (abstract) Riemann surface, one important class is formed by complex-valued functions.

Let M^2 be an (abstract) Riemann surface, and $f : M^2 \to C$ a function on it assuming its values in the field C of complex numbers. The function f is said to be *regular analytic* or *holomorphic* at a point $P_0 \in M^2$ if, while being expressed in terms of the local coordinates $z = \Phi(P)$, $0 = \Phi(P_0)$ in a neighbourhood of the point P_0, it is a regular analytic function of z in a certain circle $|z| < r$, i.e.,

$$f(\Phi^{-1}(z)) = \sum_{n=0}^{\infty} a_n z^n,$$

where the power series on the right converges in the circle $|z| < r$.

A function f is said to be *analytic* in a certain open set $U \subset M^2$ if it is a regular analytic function at each point $P_0 \in U$.

Functions on the z-sphere are usually given in the local coordinates on an open set U_1 (see Item 7, Sec. 3), i.e., as functions $w = w(z)$ on the z-plane. To investigate a function in a neighbourhood of the point ∞, it is necessary to have its expression in terms of the local coordinates on the set U_2. The latter is achieved by replacing z by $1/z$: we obtain the function $w = w(1/z) = w_1(z)$ which we investigate in a neighbourhood of the origin.

Exercise 2°. Verify that the function $w = 1/z$ is defined in a neighbourhood of the point $z = \infty$ of the z-sphere and is holomorphic at this point. The same task for the function $w = \sum_{k=0}^{n} \frac{a_k}{z^k}$.

2. A Partition of Unity. The main instrument of manifold theory in transferring from local statements to global is a partition of unity.

Let M^n be a C^r-manifold, and $\{U_\alpha\}$ its open covering.

DEFINITION 4. If $\varphi : M^n \to R^1$ is a function then its *support* supp φ is the closure of the set $\{x : \varphi(x) \neq 0\}$.

DEFINITION 5. A family of C^r-functions $\{\varphi_\beta : M^n \to [0, 1]\}$ is called a *partition of unity* of class C^r subordinate to a covering $\{U_\alpha\}$ if (i) each of the sets supp φ_β is compact and contained in a certain set U_α, (ii) the family $\{\text{supp}\,\varphi_\beta\}$ forms a locally finite covering of M^n, and (iii) $\sum_\beta \varphi_\beta(x) = 1$ for any point $x \in M^n$.

The summation in condition (iii) makes sense since at each point x only a finite number of the functions φ_β is different from zero in view of condition (ii).

THEOREM 1. *For any open covering of a C^r-manifold M^n, $r = 1, \ldots, \infty$, there exists its subordinate C^r-partition of unity.*

To prove this fundamental theorem, we need several lemmata.

LEMMA 1. *For any $s \in R^1$, there exists a C^∞-function $h_s : R^1 \to [0, 1]$ such that supp $h_s \subset [s, \infty)$.*

PROOF. It is easy to verify that the function

$$h_s(x) = \begin{cases} e^{-\frac{1}{x-s}} & \text{when } x > s, \\ 0 & \text{when } x \leqslant s \end{cases}$$

is the required (Fig. 81). ∎

Exercise 3°. Construct the graphs of the functions $h_{-s}(x)$, $h_{-s}(-x)$ and $h_{s/2}(x) + h_{s/2}(-x)$.

LEMMA 2. *For any $s > 0$, there exists a C^∞-function $g_s : R^n \to [0, 1]$ such that*

$$g_s(x) = \begin{cases} 1 & \text{when } x \in \bar{D}_{s/2}(0), \\ 0 & \text{when } x \in R^n \setminus D_s(0). \end{cases}$$

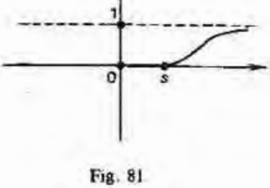

Fig. 81

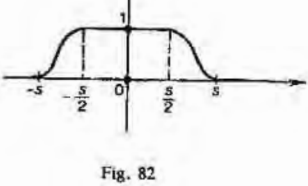

Fig. 82

PROOF Consider the function $g_s : R^1 \to [0, 1]$ (Fig. 82) given by the equality

$$g_s(x) = \frac{h_{-s}(x)h_{-s}(-x)}{h_{-s}(x)h_{-s}(-x) + h_{s/2}(x) + h_{s/2}(-x)}.$$

The function $g_s(x) = g_s(\|x\|) : R^n \to [0, 1]$ is obviously the required.

Note that if $x_0 \in R^n$ is an arbitrary point, then

$$g_s(x - x_0) = \begin{cases} 1 & \text{when } x \in \bar{D}_{s/2}(x_0), \\ 0 & \text{when } x \in R^n \setminus D_s(x_0). \blacksquare \end{cases}$$

LEMMA 3. *Let M^n be a C^r-manifold ($r = 1, \ldots, \infty$) and $U \subset M^n$ an open set. Then for any point $x_0 \in U$, there exist its open neighbourhoods $V_1(x_0)$, $V_2(x_0)$ and a C^r-function $f : M^n \to [0, 1]$, such that*

(1) $\bar{V}_1(x_0) \subset V_2(x_0) \subset U$,

(2) $f(x) = \begin{cases} 1 & \text{when } x \in \bar{V}_1(x_0), \\ 0 & \text{when } x \in M^n \setminus V_2(x_0). \end{cases}$

PROOF. Let $\{(U_\alpha, \varphi_\alpha)\}$ be a C^r-atlas on M^n and let $x_0 \in U_\alpha \cap U$. Since the set $U_\alpha \cap U$ is open, and φ_α is a homeomorphism, the set $\varphi_\alpha^{-1}(U_\alpha \cap U)$ is open in R^n and therefore there exists a disc $D_s(\varphi_\alpha^{-1}(x_0)) \subset \varphi_\alpha^{-1}(U_\alpha \cap U)$. Furthermore, since φ_α is a homeomorphism,

$$\overline{\varphi_\alpha(D_{s/2}(\varphi_\alpha^{-1}(x_0)))} = \varphi_\alpha(\bar{D}_{s/2}(\varphi_\alpha^{-1}(x_0))) \subset \varphi_\alpha(D_s(\varphi_\alpha^{-1}(x_0))) \subset U,$$

therefore, the sets $V_1(x_0) = \varphi_\alpha(D_{s/2}(\varphi_\alpha^{-1}(x_0)))$, $V_2(x_0) = \varphi_\alpha(D_s(\varphi_\alpha^{-1}(x_0)))$ satisfy condition (1). By Lemma 2, there exists a C^r-function $g_{s,x_0}(x) = g_s(x - x_0)$ such that

$$g_{s,x_0}(x) = \begin{cases} 1 & \text{when } x \in \bar{D}_{s/2}(\varphi_\alpha^{-1}(x_0)), \\ 0 & \text{when } x \in R^n \setminus D_s(\varphi_\alpha^{-1}(x_0)). \end{cases}$$

Then the function

$$f(x) = \begin{cases} g_{s,x_0}\varphi_\alpha^{-1}(x) & \text{when } x \in V_2(x_0), \\ 0 & \text{when } x \in M^n \setminus V_2(x_0) \end{cases}$$

evidently satisfies condition (2). $\blacksquare$

LEMMA 4. *Let M^n be a manifold of class C^r, $r = 1, \ldots, \infty$; $K \subset U \subset M^n$, where the set K is compact, and the set U is open. Then there exists a C^r-function $f : M^n \to [0, 1]$ such that*

$$f(x) = \begin{cases} 1 & \text{when } x \in K, \\ 0 & \text{when } x \in M^n \setminus U. \end{cases}$$

PROOF. By Lemma 3, for any point $y \in K$, there exist open neighbourhoods $V_1(y)$, $V_2(y)$ such that $\bar{V}_1(y) \subset V_2(y) \subset U$ and there is a C^r-function $f_y(x) : M^n \to [0, 1]$

such that

$$f_y(x) = \begin{cases} 1 & \text{when } x \in \bar{V}_1(y), \\ 0 & \text{when } x \in M^n \setminus V_2(y). \end{cases}$$

Due to the compactness of K, in an open covering $\{V_1(y)\}_{y \in K}$ of the set K, a finite subcovering $V_1(y_1, \ldots, V_1(y_p)$ is contained. Put

$$g(x) = \prod_{i=1}^{p} (1 - f_{y_i}(x)),$$

then

$$g(x) = \begin{cases} 0 & \text{when } x \in K, \\ 1 & \text{when } x \in M^n \setminus U, \end{cases}$$

$$f(x) = 1 - g(x) = \begin{cases} 1 & \text{when } x \in K, \\ 0 & \text{when } x \in M^n \setminus U. \end{cases}$$

It is evident that $f \in C^r$. ■

LEMMA 5. *A locally finite covering $\{U'_\beta\}$ such that each set $\bar{U}'_\beta$ is compact and contained in a certain U_α, can be made a refinement of any open covering $\{U_\alpha\}$ of a C^r-manifold M^n, $r = 1, \ldots, \infty$.*

PROOF. Let $\{(V_\nu, \varphi_\nu)\}$ be a C^r-atlas on the manifold M^n. The sets $\{V_\nu \cap U_\alpha\}$ form an open covering that refines $\{U_\alpha\}$. Since φ_ν is a homeomorphism, the set $\varphi_\nu^{-1}(V_\nu \cap U_\alpha)$ is open in R^n and therefore, for any point $x \in V_\nu \cap U_\alpha$, the point $\varphi_\nu^{-1}(x)$ is contained in the set $\varphi_\nu^{-1}(V_\nu \cap U_\alpha)$ with a certain disc $D_{s(x)}(\varphi_\nu^{-1}(x))$. Furthermore, since φ_ν is a homeomorphism,

$$\overline{\varphi_\nu(D_{s(x)/2}(\varphi_\nu^{-1}(x)))} = \varphi_\nu(\bar{D}_{s(x)/2}(\varphi_\nu^{-1}(x))) \subset \varphi_\nu(D_{s(x)}(\varphi_\nu^{-1}(x))) \subset V_\nu \cap U_\alpha; \quad (1)$$

moreover, since the set $\bar{D}_{s(x)/2}(\varphi_\nu^{-1}(x))$ is compact, and M^n Hausdorff, $\varphi_\nu(\bar{D}_{s(x)/2}(\varphi_\nu^{-1}(x)))$ is compact (see Sec. 13, Ch. II). In view of the paracompactness of the manifold M^n (see Sec. 3), a locally finite covering $\{U'_\beta\}$ can refine the open covering $\{\varphi_\nu(D_{s(x)/2}(\varphi_\nu^{-1}(x)))\}$ of M^n. Then each U'_β is contained in a certain $\varphi_\nu(D_{s(x)/2}(\varphi_\nu^{-1}(x)))$. Since

$$\bar{U}'_\beta \subset \overline{\varphi_\nu(D_{s(x)/2}(\varphi_\nu^{-1}(x)))},$$

we derive from (1) that

$$\bar{U}'_\beta \subset \varphi_\nu(D_{s(x)}\varphi_\nu^{-1}(x))) \subset V_\nu \cap U_\alpha,$$

therefore $\bar{U}'_\beta \subset U_\alpha$. The compactness of $\bar{U}'_\beta$ follows from the fact that $\bar{U}'_\beta$ is a closed subset of the compact space $\varphi_\nu(\bar{D}_{s(x)/2}(\varphi_\nu^{-1}(x)))$ (see Sec. 13, Ch. II). ■

THE PROOF OF THEOREM 1. Applying Lemma 5 twice, we take a locally finite covering $\{U'_\beta\}$ which is a refinement of the given covering $\{U_\alpha\}$, and an open,locally finite covering $\{U''_\gamma\}$ which is a refinement of $\{U'_\beta\}$ so that each $\bar{U}''_\gamma$ may be compact and contained in a certain U'_β, and each $\bar{U}'_\beta$ compact and contained in a certain U_α. For each U''_γ, we fix one set containing $\bar{U}''_\gamma$ of the system U'_β and redesignate it by U'_γ.

Now, by applying Lemma 4 to the set $K = \bar{U}_\gamma''$, we find the corresponding C^r-function $f_\gamma : M^n \to [0, 1]$ such that

$$f_\gamma(x) = \begin{cases} 1 & \text{when} \quad x \in \bar{U}_\gamma'' \\ 0 & \text{when} \quad x \in M^n \setminus U_\gamma'. \end{cases}$$

Consider the function

$$\varphi_\gamma(x) = \frac{f_\gamma(x)}{\sum\limits_\nu f_\nu(x)}, \; x \in M^n$$

(the summation in the denominator is performed over the range of indices of the covering $\{U_\gamma''\}$ and has meaning since the covering $\{U_\gamma'\}$ is locally finite and therefore at each point $x \in M^n$, only a finite number of the functions f_ν is different from zero). Since $\operatorname{supp} \varphi_\gamma = \operatorname{supp} f_\gamma$ and the set $\operatorname{supp} f_\gamma$ is compact (as a closed subset of the compact space $\bar{U}_\gamma'$), the set $\operatorname{supp} \varphi_\gamma$ is also compact. It is easy to see that $\varphi_\gamma \in C^r$, and thus we have obtained the required partition of unity. ■

DEFINITION 6. A function $f : A \to R^1 (A \subset M^n)$ is called a *C^r-function* on a set A if it is a restriction to A of a C^r-function given on an open subset U of the manifold M^n, which contains A.

Exercise 4°. Using the unity partition theorem, prove that Definition 3 of a smooth function given on a set $A \subset M^n$ is equivalent to Definition 6.

3. **The Algebra of C^r-functions on a Manifold.** Consider now the set $\mathcal{O}(M^n)$ of all C^r-functions on a C^r-manifold M^n. The functions from $\mathcal{O}(M^n)$ may be added together and multiplied by real numbers in a natural fashion: if f, $g \in \mathcal{O}(M^n)$, $\alpha \in R^1$, then for each point $x \in M^n$, we put $(f + g)(x) = f(x) + g(x)$ and $(\alpha f)(x) = \alpha f(x)$. Thus, $\mathcal{O}(M^n)$ has been transformed into a vector space. Moreover, usual multiplication $(f \cdot g)(x) = f(x) \cdot g(x)$, $x \in M^n$ transforms $\mathcal{O}(M^n)$ into an algebra over the field R.

Let x be a certain point of the manifold M^n. Consider the following equivalence relation on the algebra $\mathcal{O}(M^n)$: $f_1 \sim f_2$ if the point x possesses a neighbourhood $U(x)$ such that $f_1|_{U(x)} = f_2|_{U(x)}$. Call the equivalence class the *C^r-germ* (of the functions) at the point x, and denote the totality of all C^r-germs at the point x by $\mathcal{O}(x)$. It is obvious that $\mathcal{O}(x)$ is also an algebra. We now give another definition of $\mathcal{O}(x)$.

If the difference algebra $\mathcal{O}(M^n)/\mathcal{O}_0(x)$ is considered, where $\mathcal{O}_0(x)$ is the ideal of all those functions from the ring $\mathcal{O}(M^n)$ which assume the zero value in a certain (depending on the function) neighbourhood of the point x, then its elements can be naturally identified with the germs of the functions at the point x. It is easy to see that $\mathcal{O}(M^n)/\mathcal{O}_0(x) = \mathcal{O}(x)$. The set of germs $\mathcal{O}(x)$ could also be defined as the set of C^r-functions in the neighbourhoods of the point x and which is factorized with respect to the same equivalence relation as in the definition of $\mathcal{O}(x)$. Prima facie, a new object is obtained since we consider functions defined only on a part of the manifold M^n. It follows from the following exercise that this is not correct.

Exercise 5°. Let f be a function of class C^r defined in an open neighbourhood $U(x)$ of a point x of a manifold M^n of class C^r. Show that there exist a closed

neighbourhood $\bar{V}(x)$ of the point x, $\bar{V}(x) \subset U(x)$ and a C^r-function $\bar{f}$ defined in the whole manifold M^n such that $\bar{f}|_{\bar{V}(x)} = f_{\bar{V}(x)}$.

Hint: Use Lemma 4.

Thus, the algebraic structures of the smooth functions $\mathscr{O}(M^n)$ and germs $\mathscr{O}(x)$ have been constructed on a smooth manifold. An interesting question arises whether it is possible, conversely, by means of the algebras $\mathscr{O}(M^n)$ and germs $\mathscr{O}(x)$, to restore the structure of the manifold. We show below that this can be done.

First of all, we fix, axiomatically, the most essential properties of the algebras of functions in a smooth manifold. Consider a topological space M and real functions $f, f_1, \ldots, f_k$ defined on M. We will say that f *C^r-smoothly depends on the functions* $f_1, \ldots, f_k (r \geqslant 1)$ if there exists a C^r-function $U(t_1, \ldots, t_k)$ of real variables $t_1, \ldots, t_k$ defined on R^k such that

$$f(x) = U(f_1(x), \ldots, f_k(x)), x \in M. \tag{2}$$

If equality (2) is valid only for points of a certain set $V \subset M$ then we will say that the function f smoothly depends on the functions $f_1, \ldots, f_k$ on the set V. Call a *C^r-smoothness* on a topological space M a nonempty set $\mathscr{S}(M)$ of real functions on M, which satisfies the conditions:

(i) any function C^r, which smoothly depends on the functions from $\mathscr{S}(M)$, belongs to $\mathscr{S}(M)$,

(ii) any function on M, which coincides with a function from $\mathscr{S}(M)$ in a neighbourhood of each point $x \in M$, belongs to $\mathscr{S}(M)$.

Exercise 6°. Verify that for a C^r-manifold M^n, the algebra of C^r-functions $\mathscr{O}(M^n)$ satisfies the conditions for a C^r-smoothness.

It follows from condition (i) that the set $\mathscr{S}(M)$ is an algebra under the natural operations of addition and multiplication of functions and their multiplication by a number. The notions of C^r-germ $\hat{f}_x$ of a function $f \in \mathscr{S}(M)$ at a point x, of the ideal $\mathscr{S}_0(x)$ and of the set of C^r-germs $\mathscr{S}(x) = \mathscr{S}(M)/\mathscr{S}_0(x)$ are defined in a natural manner.

Now we take up the construction of the C^r-structure on M. Let M be a topological space with a C^r-smoothness $\mathscr{S}(M)$. Let the following conditions be fulfilled: (i) for any point $x \in M$, there are germs $\hat{f}_x^1, \ldots, \hat{f}_x^n \in \mathscr{S}(x)$, a neighbourhood $V(x)$, representatives $f^i : (V(x) \to R^1, i = 1, \ldots, n$, of the germs $\hat{f}_x^i$ such that the mapping $\psi_V : y \to [f^1(y), \ldots, f^n(y)], y \in V(x)$ is a homeomorphism of $V(x)$ onto the space R^n, (ii) for any point $y \in V(x)$, the germs $\hat{f}_y^1, \ldots, \hat{f}_y^n$ of the functions $f^1, \ldots, f^n$ belong to $\mathscr{S}(y)$; (iii) for any germ $\hat{g}_y \in \mathscr{S}(y)$, its representative g C^r-smoothly depends on $f^1, \ldots, f^n$ in a neighbourhood of the point y. Thus, specifying a coordinate system in a neighbourhood $V(x)$ by means of the homeomorphism $\varphi_V = \psi_V^{-1} : R^n \to V(x)$, we obtain a system of charts $\{(V(x), \varphi_V)\}$, which, as can be easily verified with the help of properties (ii) and (iii), produces a C^r-atlas on M.

Exercise 7°. Show that the system of charts $\{(V(x), \varphi_X)\}$ yields an atlas.

Thus, the differential structure of a C^r-manifold induced by the algebras $\mathscr{S}(M)$, $\mathscr{S}(x)$ has been defined on M.

Exercise 8°. Show that if M^n is a C^r-manifold and $\{\mathscr{S}(x)\}_{x \in M^n}$ are the corresponding algebras of the germs of the C^r-functions on M^n, then the differential structure determined by them coincides with the structure of the manifold M^n.

NOTE. Conditions (i) and (iii) make us conclude that the considered smoothness on M consists of continuous functions. We could also consider C^r-smoothness on an abstract set M and induce the weakest topology on it so that all the functions from the smoothness might be continuous.

5. MAPPINGS OF MANIFOLDS

1. The Notion of Smooth Mapping. Let us define and investigate smooth mappings of smooth manifolds, which are a natural generalization of differentiable functions considered in analysis. Let M^n, N^m be C^r-manifolds, $r \geqslant 1$. Regarding M^n, N^m as topological spaces, we may speak of continuous mappings $f: M^n \to N^m$. The structures of class C^r, given on M^n, N^m, admit introducing a narrower class of mappings. The mapping $f: M^n \to N^m$ can be specified naturally in local coordinates. Viz., if $x \in M^n$ is an arbitrary point, (U, φ) and (V, ψ) are charts on the manifolds M^n, N^m, respectively, such that $x \in U$, $f(x) \in V$, and $W(x)$ is an open neighbourhood of the point x such that $W(x) \subset U$, $f(W(x)) \subset V$, then the mapping

$$\psi^{-1}f\varphi : \varphi^{-1}(W(x)) \to \psi^{-1}(V)$$

is called the *coordinate representation of the mapping* f in a neighbourhood of the point x. Such a representation enables us to involve the notion of smooth mapping of R^n to R^m which is studied in analysis (see Sec. 1).

DEFINITION 1. A mapping $f: M^n \to N^m$ is called a *C^r-mapping (a mapping of class C^r) in a neighbourhood of a point $x \in M^n$* if some coordinate representation of the mapping f in the neighbourhood of the point x is a C^r-mapping.

Exercise 1°. Show that the definition of a C^r-mapping in a neighbourhood of a point does not depend on the choice of a coordinate representation.

In defining a smooth mapping in a neighbourhood of a point, it is natural to consider mappings defined not in the whole of M^n, but in an open neighbourhood of the point.

Definition 1 can be restated in other terms for the case of submanifolds in R^N Let M^n, N^m be submanifolds in R^{N_1} and R^{N_2}, respectively.

DEFINITION 2. A mapping $f: M^n \to N^m$ is called a *C^r-mapping (a mapping of class C^r) in a neighbourhood of a point $x \in M^n$* if there exist an open set $U \subset R^{N_1}$, $x \in U$ and a C^r-mapping $\bar{f}: U \to R^{N_2}$ that coincides with f on $U \cap M^n$.

Exercise 2°. Show that Definitions 1 and 2 are equivalent for the case of submanifolds in R^N.

Hint: Use the property of mappings of charts to be diffeomorphisms (see Lemma 1, Sec. 2).

We now come over from local definitions to global.

DEFINITION 3. A mapping $f : M^n \to N^m$ of manifolds is called a C^r*-mapping* (*a mapping of class* C^r) if it is a C^r-mapping in a neighbourhood of each point $x \in M^n$.

It is evident that the notion of C^r-mapping is a generalization of the notion of C^r-function.

Similarly, the notion of complex analytic function on a Riemann surface is generalized into the notion of complex analytic mapping of Riemann surfaces (if we require the coordinate representation to be analytic).

Exercises.

3°. Verify that the mapping $w = \sqrt{z}$ of a two-sheeted Riemann surface onto the z-sphere is analytic.

4°. Verify that the mappings $w = 1/z$ and $w = \sum_{k=0}^{n} \frac{a_k}{z^k}$ considered as mappings of the z-sphere onto themselves are analytic.

DEFINITION 4. A mapping $f : M^n \to N^n$ of manifolds of class C^r is called a C^r-*diffeomorphism* if (i) f is bijective, (ii) f, f^{-1} are C^r-mappings.

Exercise 5°. Why cannot a diffeomorphism of manifolds of different dimensions be defined?

Two C^r-manifolds M^n, N^n are said to be C^r*-diffeomorphic* if there exists a C^r-diffeomorphism $f : M^n \to N^n$.

Exercise 6°. Show that the collection of n-dimensional C^r-manifolds $\{M^n\}$, $n \geq 1$, forms a category whose morphisms are C^r-mappings of manifolds. Show that the equivalences on this category are C^r-diffeomorphisms of manifolds.

THEOREM 1. *If M^n is a manifold of class C^r and N^n a manifold of class C^r with the structure induced by the homeomorphism $f : M^n \to N^n$, then M^n and N^n are C^r-diffeomorphic.*

PROOF. It is easy to see that the required diffeomorphism is f. ∎

Thus, by inducing the structure of a C^r-manifold on a topological space N via the homeomorphism $f : M^n \to N$, we transform f into a C^r-diffeomorphism.

THEOREM 2. *Let $f : M^n \to N^n$ be a C^r-diffeomorphism of C^r-manifolds M^n, N^n. Then the mapping $f : M^n \to N^n$ as a homeomorphism of topological spaces induces on N^n a C^r-structure coinciding with the original.*

PROOF. Let $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ be C^r-atlases on M^n and N^n, respectively. We show that any chart of the atlas $\{(f(U_\alpha), f\varphi_\alpha\}$ is C^r-compatible with any chart of the atlas $\{(V_\beta, \psi_\beta)\}$, i.e., that the mapping

$$\psi_\beta^{-1}(f\varphi_\alpha) : (f\varphi_\alpha)^{-1}(f(U_\alpha) \cap V_\beta) \to \psi_\beta^{-1}(f(U_\alpha) \cap V_\beta) \tag{1}$$

is a C^r-diffeomorphism for any α and β.

In fact, since f is a C^r-diffeomorphism, its representation in the local coordinates on the open sets $f^{-1}(f(U_\alpha) \cap V_\beta) \subset U_\alpha$, $f(U_\alpha) \cap V_\beta \subset V_\beta$

$$\psi_\beta^{-1} f \varphi_\alpha : \varphi_\alpha^{-1}[f^{-1}(f(U_\alpha) \cap V_\beta)] \to \psi_\beta^{-1}(f(U_\alpha) \cap V_\beta)$$

is a C^r-diffeomorphism. But

$$\varphi_\alpha^{-1}[f^{-1}(f(U_\alpha) \cap V_\beta)] = (f\varphi_\alpha)^{-1}(f(U_\alpha) \cap V_\beta);$$

therefore, mapping (1) is a C^r-diffeomorphism, which proves the statement. ■

From the point of view of general topology, we do not make any distinction between homeomorphic spaces. It is natural to agree not to distinguish between homeomorphic manifolds M^n and N^n, where the manifold N^n is endowed with the structure of the smooth manifold induced by the homeomorphism $f: M^n \to N^n$. But then, according to Theorem 1, M^n and N^n are diffeomorphic. Conversely, if the manifolds M^n, N^n are diffeomorphic ($f: M^n \to N^n$) then they are homeomorphic and, according to Theorem 2, the homeomorphism f induces on N^n the structure of a smooth manifold which coincides with the original. Thus, the adopted agreement is equivalent to the practice not to distinguish between diffeomorphic manifolds.

Exercise 7°. Verify that being diffeomorphic is an equivalence relation for manifolds.

A question naturally arises whether there exist homeomorphic but not diffeomorphic manifolds. This question was resolved by Milnor who showed that there are exactly 28 smooth manifolds (or the Milnor spheres) which are homeomorphic to S^7 but not diffeomorphic to each other*. It is known also that if the dimension of a manifold is less than 4, then homeomorphity entails diffeomorphity, i.e., for a manifold of dimension less than 4, differentiability and topology classifications coincide.

2. Regular and Nonregular Points of a Smooth Mapping. Immersions, Submersions, Embeddings and Submanifolds.

Let $f: M^n \to N^m$ be a C^r-mapping of C^r-manifolds, $r \geqslant 1$.

DEFINITION 5. A point $x \in M^n$ is called a *regular* (*noncritical*, or *nonsingular*) *point* of a mapping f if for a certain coordinate representation

$$\psi^{-1}f\varphi : \varphi^{-1}(W(x)) \to \psi^{-1}(V)$$

of the mapping f in a neighbourhood of the point x, the point $\varphi^{-1}(x)$ is regular.

Otherwise the point x is said to be *nonregular* (*critical*, or *singular*).

Exercise 8°. Show that the definition is independent of the choice of a coordinate representation.

DEFINITION 6. A point $y \in N^m$ is said to be *regular* (*noncritical*, or *nonsingular*) *value* of a mapping f if its full inverse image $f^{-1}(y)$ either consists of only regular points of the mapping f or is empty.

Otherwise the point y is called a *nonregular* (*critical*, or *singular*) *value*.

Let for a C^r-mapping $f: M^n \to N^m$ of C^r-manifolds, each point $x \in M^n$ be regular. Such a mapping is called (i) a *C^r-immersion* when $n \leqslant m$, (ii) a C^r-

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* The Milnor spheres can be given as submanifolds in $R^{10} = C^5 = \{(z_1, \ldots, z_5)\}$ by the systems of two equations $z_1^{6k-1} + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0$, $|z_1|^2 + \ldots + |z_5|^2 = 1$, $k = 1, 2, \ldots, 28$

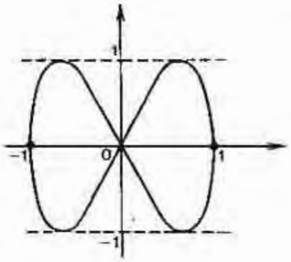

Fig. 83

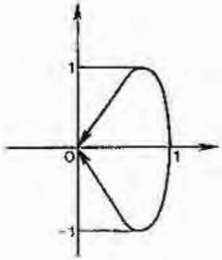

Fig. 84

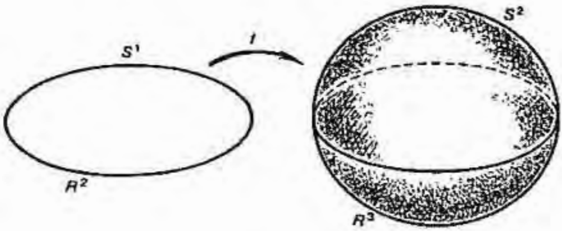

Fig. 85

submersion when $n \geqslant m$, (iii) a C^r-immersion is called a *C^r-embedding* if f is a homeomorphism of M^n onto the subspace $f(M^n)$ of the topological space N^m and is usually denoted by i.

EXAMPLES.

1. The mapping $f: R^2 \to R^1$ given by the rule $f(x, y) = x$ is a C^∞-submersion.
2. The mapping $f: R^1 \to R^2$ given by the rule $f(x) = (\sin x, \sin 2x)$ (Fig. 83) is a C^∞-immersion but not an embedding since it is not injective. The mapping $f|_{(0, 2\pi)}$ is not an embedding either, though f is bijective on $(0, 2\pi)$. In this case, the mapping f^{-1} is not continuous. Note that the mapping $f|_{(0, \pi)}$ (Fig. 84) is a C^∞-embedding.
3. The mapping $f: S^1 \to S^2$ given by the rule $f(x, y) = (x, y, 0)$ (Fig. 85) is a C^∞-embedding.
4. Let $f_1, f_2: R^1 \to R^1$ be functions of class C^r. The mapping $f = (f_1, f_2): R^1 \to R^2$ (the curve is of class C^r) can be considered to be a C^r-mapping of manifolds R^1, R^2 with the natural C^∞-structure. Elucidate the conditions under which the mapping f is an immersion. Immersion implies that any point $x \in R^1$ is a regular point of the mapping

$$(1_{R^2})^{-1} f 1_{R^1} = f: R^1 \to R^2$$

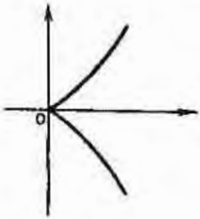

Fig. 86

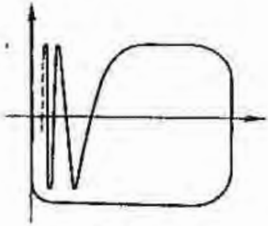

Fig. 87

(here 1_{R^2}, 1_{R^1} are the identity mappings of R^2 and R^1), i.e.,

$$\operatorname{rank}\left(\frac{df_1}{dx},\ \frac{df_2}{dx}\right) = 1. \tag{2}$$

Thus, f is a C^r-immersion if all the derivatives $\frac{df_1}{dx}$, $\frac{df_2}{dx}$ are never zero simultaneously.

A curve satisfying condition (2) is called a curve without singular points. Those points at which condition (2) is not fulfilled are called *singular points* of the curve. E.g., for the curve $f_1(x) = x^2, f_2(x) = x^3$ (Fig. 86), the point 0 is singular.

5. The curve drawn in Fig. 87 (constructed by means of the graph of the function $y = \sin\frac{1}{x}$) determines a C^∞-immersion, but not an embedding of the half-line into the plane though the mapping is bijective.

Another example of a similar kind is given by the immersion $f:R^1 \to C \times C$ determined by the formula $f(x) = (e^{2\pi i\alpha_1 x}, e^{2\pi i\alpha_2 x})$, where $\frac{\alpha_1}{\alpha_2}$ is irrational. It is easy to verify that this is a bijective mapping (of rank 1) and that its image lies on the torus $S^1 \times S^1$ and encircles it (being everywhere dense).

Note that the noncompactness of the straight line played an important role in the given examples. In fact, the following theorem is valid.

THEOREM 3. *If a manifold M^n is compact and $f : M^n \to N^m$ is an injective immersion, then f is an embedding.*

The proof follows from the fact that an injective, continuous mapping $f : M \to N$ of a compact space M onto $f(M) \subset N$ is a homeomorphism onto the subspace $f(M)$ (see Sec. 13, Ch. II).

Note that any C^r-immersion $f : M^n \to N^m$ is a C^r-embedding on a certain neighbourhood of each point $x \in M^n$ (this follows from the theorem on rectifying a mapping, see Sec. 1).

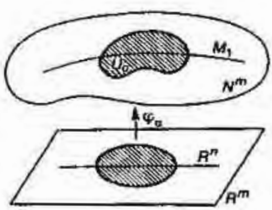

Fig. 88

EXAMPLE 6. A mapping $f : R^n \to R^N$, $N \geqslant n$, of class C^r, $r \geqslant 1$, determines an immersion in R^N if

$$\operatorname{rank} \left(\frac{\partial f}{\partial x}\right)\bigg|_y = n \tag{3}$$

at any point $y \in R^n$. Thus, f possesses no nonregular points and, by the theorem on rectifying a mapping, is a local homeomorphism between R^n and $f(R^n)$. If, in addition, f is a homeomorphism of R^n onto $f(R^n)$ then f is a C^r-embedding.

Exercise 9°. Verify that the notion of chart on a C^r-submanifold in R^N (see Sec. 2) is equivalent to the C^r-embedding of R^n in R^N.

Very often manifolds lie in other ambient manifolds. It will be too general to call any such manifold a submanifold of the ambient manifold just like a subset endowed, in a topological space, with an arbitrary topology will not be termed a subspace. It is necessary to impose reasonable restrictions if we require that there exist a simple relation between the structures of an embedded and ambient manifolds. Meanwhile, the notion of embedding proves useful.

DEFINITION 7. We call any subspace M_1 in N^m which is the image of a certain C^r-embedding $f : M^n \to N^m$ with the C^r-structure induced by the homeomorphism f, a *submanifold* of the C^r-manifold N^m.

The submanifold and manifold structures happen to be related in the following simple fashion: for a certain atlas $[(U_\alpha, \varphi_\alpha)]$ of a manifold N^m, the intersection $U_\alpha \cap M_1^n$ is (if nonempty) the image of the subspace $R^n \subset R^m = R^n \times R^{m-n}$ under the homeomorphism φ_α, the restrictions $\varphi_\alpha|_{R^n} : R^n \to U_\alpha \cap M_1^n$ determining an atlas on the submanifold ${M^n}_1$.

Thus, a submanifold in N^m is given locally in the corresponding local coordinates $\xi_1, \ldots, \xi_m$ on the manifold N^m by the equations $\xi_{n+1} = 0, \ldots, \xi_m = 0$.

Exercise 10°. By applying the theorem on rectifying to the coordinate representation of the embedding f, construct the above atlases on the manifolds N^m and M_1^n.

This interrelation of the manifold and submanifold structures may be assumed to be the basis for the notion of submanifold.

DEFINITION 8. A subspace $M_1 \subset N^m$ is called an n-dimensional *submanifold* of a C^r-manifold N^m, $n \leqslant m$, if, in a given structure of the manifold N^m, there exists a family of charts $\{(U_\alpha, \varphi_\alpha)\}$, $\varphi_\alpha : R^n \times R^{m-n} \to U_\alpha$ such that $\varphi_\alpha(R^n) = U_\alpha \cap M_1$ when $U_\alpha \cap M_1 \neq \varnothing$ and $M_1 \subset \bigcup_\alpha U_\alpha$. Moreover, the mappings $\varphi_\alpha|_{R^n} : R^n \to U_\alpha \cap M_1$ determine a C^r-atlas specifying the structure of an n-dimensional C^r-manifold on M_1 (Fig. 88). Such a structure on the manifold $M_1 \subset N^m$ is called a *structure compatible with the structure of the manifold* N^m, or simply the *structure of a submanifold*.

The equivalence of Definitions 7 and 8 is obvious.

Exercises.
11°. Verify that $\{(U_\alpha \cap M_1, \varphi_\alpha|_{R^n})\}$, $(U_\alpha \cap M_1 \neq \varnothing)$ is a C^r-atlas on M_1.
12°. Let M^n be a C^r-submanifold in $R^N = R^n \times R^{N-n}$ (see Sec. 2) and $(U(x), \varphi)$ the chart at a point $x \in M^n$. Show that (i) there exists a C^r-diffeomorphism $\tilde{\varphi} : R^N \to \tilde{U}(x)$ from the space R^N onto a certain, open in R^N, neighbourhood $\tilde{U}(x)$ of the point x, such that

$$\tilde{\varphi}|_{R^n} = \varphi, \tag{4}$$

(ii) the set of charts

$$\{(\tilde{U}(x), \tilde{\varphi})\}_{x \in M^n} \cup (R^N, 1_{R^N}) \tag{5}$$

forms a C^r-atlas on R^N (in the sense of Definition 2, Sec. 3).

It follows from Exercise 12 that R^N with atlas (5) is a C^r-manifold, and M^n, due to (4), is its submanifold. (This justifies the term a 'C^r-submanifold in R^N' given in Sec. 2. However, to be more precise, the term a 'submanifold of a C^r-manifold R^N' should be used instead.)

EXAMPLE 7. The equator of the sphere S^2 (See Example 3) is a submanifold.

Exercise 13°. Show that the graph of the mapping $f(x) = |x|$, $x \in R^1$ is not a submanifold of R^2.

Submanifolds often emerge not as images under certain mappings, but as inverse images. The following important theorem happens to be useful not only in constructing new manifolds but also often facilitates the proof of the fact that the spaces under investigation possess the manifold structure.

THEOREM 4. *Let* $f : M^n \to N^m$ $(n \geqslant M)$ *be a* C^r-*mapping of* C^r-*manifolds* $(r \geqslant 1)$, N_1^k *a submanifold in* N^m *consisting of only regular values of the mapping* f. *Then* $M_1 = f^{-1}(N_1^k)$ *is either empty or a submanifold in* M^n *of dimension* $n - m + k$.

PROOF. Assume that $M_1 \neq \varnothing$. Let x_0 be an arbitrary point in M_1. Since N_1^k is a submanifold, there exists a chart $(\tilde{V}, \psi)$, $(f(x_0) \in \tilde{V})$ from the maximal atlas for the C^r-structure given on N^m such that the pair $(\tilde{V} \cap N_1, \psi|_{R^k})$ is a chart of the maximal atlas for the C^r-structure on N_1^k. Let (U, φ), $(x_0 \in U)$ be a chart of the maximal atlas for the C^r-structure given in M^n such that $f(U) \subset \tilde{V}$. Then, from the data given, $\varphi^{-1}(x_0)$ is a regular point of the mapping $\Phi = \psi^{-1} f \varphi : \varphi^{-1}(U) \to R^m$ and, by the theorem on rectifying a mapping, there exist an open neighbourhood

$V(\varphi^{-1}(x_0)) \subset R^n$ of the point $\varphi^{-1}(x_0)$, an open set $W \subset R^n$ and a C^r-diffeomorphism $F: V(\varphi^{-1}(x_0)) \to W$, such that the mapping ΦF^{-1} on the set W is the standard projection of R^n onto R^m. Note that $\varphi(V(\varphi^{-1}(x_0)))$ is an open neighbourhood in M^n of the point x_0 and the pair $(\varphi(V(\varphi^{-1}(x_0))), \varphi F^{-1})$ is a chart of the maximal atlas for the C^r-structure given on M^n. Since ΦF^{-1} is the standard projection, and the set

$$\psi^{-1} f(\varphi(V(\varphi^{-1}(x_0))) \cap M_1) \subset R^m$$

consists of points of the form $(x_1, \ldots, x_k, 0, \ldots, 0)$, the set

$$F\varphi^{-1}(\varphi(V(\varphi^{-1}(x_0))) \cap M_1) \subset R^n$$

consists of points of the form $(x_1, \ldots, x_k, 0, \ldots, 0, x_{m+1}, \ldots, x_n)$. Thus, the chart $(\varphi(V(\varphi^{-1}(x_0))), \varphi F^{-1})$ in M^n possesses the property

$$(\varphi F^{-1})(R^{n-m+k} \cap W) = \varphi(V(\varphi^{-1}(x_0))) \cap M_1.$$

(Here $R^{n-m+k} = \{x \in R^n : x_{k+1} = \ldots = x_m = 0\}$.) Such a chart can be constructed for any point $x_0 \in M_1$. This proves that M_1 is a submanifold in M^n of dimension $n - m + k$. ■

EXAMPLE 8. It follows, in particular, from Theorem 4 that the inverse image of a regular value of the mapping $f: M^n \to N^m$ is either empty or a submanifold in M^n of dimension $n - m$.

The following fundamental fact is given without proof.

THEOREM 5 (WHITNEY). *Any C^r-manifold M^n can be C^r-embedded in the Euclidean space R^{2n}.*

The theorem may be given another enunciation: '*Any manifold M^n is diffeomorphic to a submanifold of the Euclidean space R^{2n}*'.

Since we agreed not to distinguish between diffeomorphic manifolds, it may be seen from the latter theorem that the abstract notion of manifold is not wider than that of submanifold in Euclidean spaces, and we could confine ourselves to their consideration only. However, this is not always expedient. Many problems leading to manifolds can be solved by a simpler method without involving an embedding.

3. The Sard Theorem. The Notion of the Degree Modulo 2 of a Mapping. The fundamental theorem about the 'quantity' of nonregular values of a smooth manifold is used in analysis quite often.

THEOREM 6 (SARD). *Let $f: M^n \to N^m$ be a C^r-mapping of C^r-manifolds. If $r \geq \max(n - m, 0) + 1$, then the nonregular values of the mapping f form a set of measure zero in N^m.*

Omitting quite a complicated proof, we make it clear that a subset A of a C^r-manifold N^m has measure zero ($\operatorname{mes} A = 0$) if $A = \bigcup_{k=1}^{\infty} A_k$, where each A_k is contained in a set U_k of some chart (U_k, φ_k) from the atlas on N^m and $\varphi_k^{-1}(A_k)$ has measure zero in R^m. Thus, the Sard theorem states that there is a 'relatively small' quantity of nonregular values of the smooth mapping $f: M^n \to N^m$.

Exercise 14°. Verify that the set of regular values is everywhere dense in N^m.

In conclusion, we introduce the concept of the degree modulo 2 of a mapping. This important characteristic of mappings proves to be quite useful in applications.

Let $f: M^n \to N^n$ be a C^r-mapping of C^r-manifolds ($r \geqslant 1$). Moreover, let N^n be connected, and f proper*. If $y \in N^n$ is a regular value of the mapping f then $f^{-1}(y)$ is a submanifold in M^n of dimension zero or empty. Since f is proper, the submanifold $f^{-1}(y)$ is compact and therefore consists of a finite number of points $k(y)$ ($k = 0$ if $f^{-1}(y) = \varnothing$). Using the Sard theorem, it can be shown that the residue class mod 2 of the number $k(y)$ does not depend on the choice of a regular value $y \in N^n$ of the mapping f. This residue class is called the *degree mod 2 of the mapping f* and denoted by $\deg_2(f)$.

We indicate the simplest application of this notion.

THEOREM 7. *If $\deg_2(f) \neq 0$ then the mapping f is surjective.*

PROOF. Consider an arbitrary $y_0 \in N^n$. If $y_0 \in f(M^n)$ then, obviously, $f^{-1}(y_0) \neq \varnothing$. If $y_0 \bar{\in} f(M^n)$ then y_0 is a regular value of the mapping f, and moreover $f^{-1}(y_0) = \varnothing$. Therefore, $k(y_0) = 0$ and $\deg_2(f) = 0$. ∎

6. TANGENT BUNDLE AND TANGENTIAL MAP

1. The Idea of a Tangent Space. To investigate smooth mappings further it is necessary to construct an analogue of the differential of a function, i.e., of a concept which is widely used in analysis. A tangential map defined and studied in this section is such a generalization. But it is necessary to generalize the notion of tangent to a curve (and of tangent plane to a surface) at first. The necessity of such a generalization is also caused by applications of the notion of manifold in mechanics and physics. As it was mentioned in Sec. 3, the configuration space of a mechanical system is, as a rule, a smooth manifold. Each point of this manifold is a certain position of a mechanical system. Under the action of forces, the mechanical system alters its position. The point of the configuration space corresponding to it moves describing a certain trajectory, viz., a path on a smooth manifold. An important characteristic of this motion is velocity which changes with time, generally speaking. The *state of a mechanical system* at each given moment of time is the pair (x, v), where x is the point of the manifold corresponding to the position of the system at the moment under consideration, and v is the displacement velocity of the point x. The collection of all states of a mechanical system is called *phase space*.

A question naturally arises what mathematical concept can be associated with the physical notion of velocity and, moreover, how to describe the notion of phase space with mathematical rigour. The solution of this question is prompted by the simplest physical examples. Thus, when a point mass moves along a curve, its velocity can be interpreted as a certain vector which is tangent to the curve and directed along the path of motion. If a point mass moves across a two-dimensional surface then its velocity is interpreted as a certain vector tangent to the given surface

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* A mapping $f: X \to Y$ of one topological space to another is said to be *proper* if the inverse image $f^{-1}(K)$ of any compact set $K \subset Y$ is compact in X.

and the path itself. The set of all possible velocities at a given point of a curve (resp. surface) is thus a tangent straight line (resp. tangent plane). The set of all possible velocities admissible at a given position of a mechanical system can be naturally interpreted in a similar way, i.e., as a certain tangent vector space at the corresponding point of a smooth manifold for the case of a general configuration space.

2. The Notion of Tangent Space to a Manifold.

Before giving a precise definition of this notion, it should be noted that we will now take into account the fact whether we consider the n-dimensional Euclidean space as a metric space (under the Euclidean metric) or endow it additionally, with the vector space structure. In the former case, the elements of R^n are said to be points, and vectors in the latter; we will also call R^n a vector space (previously, we did not make any distinction between these notions). Thus, e.g., considering the derivative $D_{x_0}(f) : R^n \to R^m$ of a mapping $f : R^n \to R^m$ at a point x_0 (see Sec. 1, Ch. IV), it should be emphasized that it carries out a linear mapping of vector spaces. Let R^n be a subspace of R^N. We call the pair $(x, v) \in R^N \times R^n$, where x is a point and v is a vector, a *vector v at a point x* (a *vector 'marked off' from the point x*, a *vector with the 'origin' at the point x*). Let x be an arbitrary point in R^N. We will call the collection of all vectors $v \in R^n$ marked off from a point x; the *space R^n marked off from the point x*. This collection possesses the natural structure of the n-dimensional Euclidean space which we will denote by R^n_x.

Consider now a smooth submanifold M^n in the Euclidean space R^N (see Sec. 2).

DEFINITION 1. Let M^n be a C^r-submanifold in R^N ($r \geqslant 1$), $x \in M^n$ an arbitrary point. Let (U, φ) be a chart on M^n, $x \in U$. The *tangent space* T_xM^n to the manifold M^n at a point x is the subspace marked off from the point x, i.e., the image of the vector space R^n under the mapping $D_{\varphi^{-1}(x)}\varphi : R^n \to R^N$.

Recall that the linear mapping $D_{\varphi^{-1}(x)}\varphi$ is given by the Jacobian matrix $\left(\dfrac{\partial\varphi}{\partial x}\right)\Big|_{\varphi^{-1}(x)}$. Since rank $\left(\dfrac{\partial\varphi}{\partial x}\right)\Big|_{\varphi^{-1}(x)} = n$, the tangent space is of dimension n.

We show the independence of the definition of T_xM^n from the choice of a chart. Let (V, ψ), $x \in V$ be another chart. The commutative diagram (on the left) generates the commutative diagram of linear mappings (on the right):

$$\begin{tikzcd} & U\cap V & \\ \varphi^{-1}(U\cap V) \arrow[ur, "\varphi"] \arrow[rr, "\psi^{-1}\varphi"'] & & \psi^{-1}(U\cap V) \arrow[ul, "\psi"'] \end{tikzcd} \qquad \begin{tikzcd} & R^N & \\ R^n \arrow[ur, "D_{\varphi^{-1}(x)}\varphi"] \arrow[rr, "D_{\varphi^{-1}(x)}(\psi^{-1}\varphi)"'] & & R^n \arrow[ul, "D_{\psi^{-1}(x)}\psi"'] \end{tikzcd}$$

Since $\psi^{-1}\varphi : \varphi^{-1}(U \cap V) \to \psi^{-1}(U \cap V)$ is a diffeomorphism, $D_{\varphi^{-1}(x)}(\psi^{-1}\varphi) : R^n \to R^n$ is an isomorphism and therefore, denoting the image by Im, we have $\operatorname{Im} D_{\varphi^{-1}(x)}(\psi^{-1}\varphi) = R^n$. Further, we obtain

$$\begin{aligned} \operatorname{Im} D_{\varphi^{-1}(x)}\varphi &= \operatorname{Im} D_{\varphi^{-1}(x)}[\psi(\psi^{-1}\varphi)] \\ &= \operatorname{Im}\{[D_{\psi^{-1}(x)}\psi][D_{\varphi^{-1}(x)}(\psi^{-1}\varphi)]\} = \operatorname{Im} D_{\psi^{-1}(x)}\psi, \end{aligned}$$

which proves the correctness of Definition 1.

The elements of the space T_xM^n are called the *tangent vectors* to M^n at the point x.

For a manifold M^2 in R^3, the tangent space T_xM^2 is a two-dimensional plane passing through the point x,which coincides with the tangent plane to the surface M^2, usually considered in analysis.

EXAMPLE 1. Let $U \subset R^n$ be an open set considered as a submanifold in R^n. Then for any point $x \in U$, we have $T_xU = R^n_x$. ♦

Extend the notion of tangent space to the case of arbitrary manifolds. In this case, generally speaking, it is impossible to speak of the derivative $D_{\varphi^{-1}(x)}\varphi$. However, from Definition 1 (of a tangent space), another approach can be deduced.

Let (U, φ) be a certain chart on a submanifold M^n in R^N, and $x \in U$ a point. The vector $(x, D_x\varphi^{-1}(h))$ from R^n_x is called the *coordinate representation of the tangent vector* $(x, h) \in T_xM^n$ in the chart (U, φ). A question arises how the coordinate representations of the tangent vector h in various charts are related. Let (V, ψ) be another chart, $x \in V$. Differentiating the mapping $\varphi^{-1} = (\varphi^{-1}\psi)\psi^{-1}$, we shall see that the coordinate representations of the vector (x, h) in the charts (U, φ) (V, ψ) are related by the equality

$$(x, D_x\varphi^{-1}(h)) \overset{*}{=} (x, D_{\psi^{-1}(x)}(\varphi^{-1}\psi)D_x\psi^{-1}(h)). \tag{1}$$

It is natural to identify a tangent vector with the set of all its coordinate representations. This note can be assumed as the basis for a new definition of a tangent vector which is suitable for an arbitrary manifold.

Let M^n be a manifold of class C^r, $r \geqslant 1$. Fix an arbitrary point $x \in M^n$ and consider the set T of all triples $(x, (U, \varphi), h)$, where (U, φ) is a chart at the point x, and h a vector of the space R^n. We define an equivalence relation on the set T as follows:

$$(x, (U, \varphi), h) \sim (x, (V, \psi), g) \Leftrightarrow h = D_{\psi^{-1}(x)}(\varphi^{-1}\psi)(g).$$

Exercise 1°. Verify that this relation is an equivalence relation.

The equivalence class $(x, (U, \varphi), h)$ is called the *tangent vector* at the point x, and the triple $(x, (U, \varphi), h)$ from the equivalence class a *representative of the tangent vector* in the chart (U, φ). Moreover, we will call the vector h the *vector component of the representative* $(x, (U, \varphi), h)$ and denote it by h^*_φ.

We next consider the set of all tangent vectors at a point x. Denoting it by T_xM^n, we fix a chart (U, φ), $x \in U$, and construct the mapping

$$\tau_x : T_xM^n \to R^n \tag{2}$$

that associates each tangent vector with the component h of its representative in the chart (U, φ). It is obvious that τ_x is a bijection and therefore the structure of the n-dimensional vector space R^n is naturally transferred to the set T_xM^n. A more detailed examination of this fact shows that the algebraic operations of addition and multiplication by a number are introduced on T_xM^n in terms of the corresponding operations over the vector components of the representatives of the tangent vectors

* Equality (1) demonstrates how the vector component varies with a change of the chart.

in the chosen chart (U, φ). If the representatives of the tangent vectors are given in different charts then they should be replaced beforehand by equivalent representatives in the same chart. Thus, the algebraic operations on T_xM^n are defined as follows:

(1) $[(x, (U, \varphi), h)] + [(x, (V, \psi), g)] = [(x, (U, \varphi), h + D_{\psi^{-1}(x)}(\varphi^{-1}\psi)(g))]$,
(2) $\alpha[(x, (U, \varphi), h)] = [(x, (U, \varphi), \alpha h)]$.

Exercise 2°. Prove the correctness of the definition of the algebraic operations and verify that the axioms of the vector space are fulfilled.

Thus, we related each point x of a manifold M^n to a vector space called the *tangent space* to M^n at the point x and denoted by T_xM^n.

The dimension of a tangent space at each point equals n, i.e., the dimension of the manifold M^n. In fact, this follows from bijection (2) (with the given definition of the algebraic operations) being an isomorphism of vector spaces.

We adduce here another convenient definition of a tangent space. Let M^n be a smooth manifold and $x \in M^n$ an arbitrary point. We call a smooth mapping $\chi : (a, b) \to M^n$, where (a, b) is a certain interval of the number line considered as a manifold with the natural C^∞-structure, a *smooth curve* χ on the manifold M^n.

Two curves χ_1 and χ_2 of the set of smooth curves

$$\chi : (-a, a) \to M^n, \chi(0) = x$$

are called *equivalent at a point* x if for a certain chart (U, φ) containing the point x, the curves $\varphi^{-1}\chi_1$, $\varphi^{-1}\chi_2$ in R^n possess the property

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi^{-1}\chi_1)(t)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(\varphi^{-1}\chi_2)(t)\bigg|_{t=0}.$$

Exercises.
3°. Show that the definition of the equivalence of the curves χ_1, χ_2 does not depend on the choice of a chart.
4°. Show that the equivalence of curves at a point is an equivalence relation on the set of smooth curves on a manifold.

DEFINITION 2. The equivalence class of smooth curves passing through a point x is called a *tangent vector* to the manifold M^n at the point x.

LEMMA 1. *The set of equivalence classes of smooth curves on a manifold M^n that pass through a point x is an n-dimensional vector space.*

In fact, having fixed a chart (U, φ), the class of equivalent curves at a point x may be associated with the n-dimensional vector $\alpha = \frac{\mathrm{d}}{\mathrm{d}t}(\varphi^{-1}\chi)\Big|_{t=0}$. Conversely, each vector α determines a straight line in the space R^n passing through the point $\varphi^{-1}(x)$ with the 'slope' α, while its image under the mapping φ will determine a smooth curve χ in R^n passing through the point x and such that $\alpha = \frac{\mathrm{d}}{\mathrm{d}t}(\varphi^{-1}\chi)\Big|_{t=0}$. Thus,

we have a bijective correspondence between the equivalence classes of curves at a point x and vectors of the space R^n.

Define the algebraic operations on the set of classes of curves equivalent at a point x so that this bijection may become an isomorphism of vector spaces:

(i) the *sum* $[\chi_1] + [\chi_2]$ of two classes is a class $[\chi_3]$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi^{-1}\chi_1)\bigg|_{t=0} + \frac{\mathrm{d}}{\mathrm{d}t}(\varphi^{-1}\chi_2)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(\varphi^{-1}\chi_3)\bigg|_{t=0};$$

(ii) the *product* $\lambda[\chi]$ of a number λ by a class $[\chi]$ is a class $[\chi_\lambda]$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi^{-1}\chi_\lambda)\bigg|_{t=0} = \lambda\frac{\mathrm{d}}{\mathrm{d}t}(\varphi^{-1}\chi)\bigg|_{t=0}.$$

Exercise 5°. Show the validity of the introduced operations and verify that the axioms of the vector space are fulfilled. ■

The n-dimensional vector space of classes of equivalent curves at a point x on a manifold M^n constructed above is called the *tangent space* to M^n at the point x, and its elements are called *tangent vectors*. It is still denoted by T_xM^n. Note that for such a definition of a tangent space, the isomorphism $\tau_x : T_xM^n \to R^n$ corresponding to the chart (U, φ), $x \in U$, is given by the formula

$$[\chi] \to \frac{\mathrm{d}}{\mathrm{d}t}(\varphi^{-1}\chi)\bigg|_{t=0}.$$

It is natural to call the triple $\left(x, (U, \varphi) \frac{\mathrm{d}}{\mathrm{d}t}(\varphi^{-1}\chi)\Big|_{t=0}\right.$ the *representative of the tangent vector* $[\chi]$ in the chart (U, φ) and the vector $\frac{\mathrm{d}}{\mathrm{d}t}(\varphi^{-1}\chi)\Big|_{t=0}$ the *vector component of the representative*.

3. **Tangent Bundle.** The tangent space T_xM^n can be defined for any point x of a smooth manifold M^n. Our next problem is to construct a topological space and even a smooth manifold from all vectors of this family of vector spaces that depend on the point x.

Considering the disjoint union $TM^n = \cup T_xM^n$ of all tangent spaces to a manifold M^n, we define the projection $\pi : TM^n \to M^n$ by mapping each vector from T_xM^n into the point x. Then

$$\pi^{-1}(x) = T_xM^n.$$

This inverse image is called the *fibre* over the point x.

Each chart (U, φ) of a manifold M^n will define the chart $(\pi^{-1}(U), \tau_\varphi)$ in TM^n

$$\tau_\varphi : \pi^{-1}(U) \to R^n \times R^n \tag{3}$$

as follows: assign to the tangent vector $a = [(x, (U, \varphi), h)]$ in the fibre over a point $x \in U$ a pair $(\varphi^{-1}(x), \tau_x a)$, where τ_x is as defined earlier (see Item 2), i.e.,

$$\tau_\varphi a = (\varphi^{-1}(x), h_\varphi),$$

$(x, (U, \varphi), h_\varphi)$ being a representative of the vector a in the chart (U, φ). It is obvious that τ_φ is bijective, therefore the weakest topology can be introduced on $\pi^{-1}(U)$ so that τ_φ may become a continuous mapping and even a homeomorphism (see Sec. 8, Ch. II). Since the set of all charts on $\pi^{-1}(U)$ forms a covering of TM^n, by declaring the collection of all open sets in all charts on $\pi^{-1}(U)$ to be the base for the topology, we thereby construct a topology on TM^n and convert TM^n into a topological space.

NOTE. According to the formal definition, the pairs $(\pi^{-1}(U), \tau_\varphi^{-1})$ should have been called charts on the space TM^n. We reversed the homeomorphisms for convenience (verify that this change is inessential).

Thus, chart (3) enables us to introduce local coordinates on the set $\pi^{-1}(U)$ by specifying the coordinates of the pair $(\varphi^{-1}(x), h_\varphi)$ in $R^n \times R^n$. We will call the pair $(\varphi^{-1}(x), h_\varphi)$ the *coordinate representation* of the tangent vector a in the chart (U, φ), and the vector h_φ the *vector component* (in the chart (U, φ)) of the tangent vector.

This terminology is justified by the following statement.

LEMMA 2. *If M^n is manifold of class C^r, $r \geqslant 1$, then the collection $\{(\pi^{-1}(U), \tau_\varphi)\}$ of all charts on the space TM^n is a C^{r-1}-atlas.*

PROOF. Let (U, φ), (V, ψ) be two charts on a manifold M^n, $U \cap V = \varnothing$. Let $(\pi^{-1}(U), \tau_\varphi)$, $(\pi^{-1}(V), \tau_\psi)$ be the corresponding charts on TM^n; then $(\pi^{-1}(U) \cap \pi^{-1}(V)) = \pi^{-1}(U \cap V) \neq \varnothing$. We have the commutative diagram

$$\begin{tikzcd} & \pi^{-1}(U \cap V) \arrow[dl, "\tau_\varphi"'] \arrow[dr, "\tau_\psi"] & \\ R^n \times R^n \arrow[rr, "\tau_\psi \tau_\varphi^{-1}"'] & & R^n \times R^n \end{tikzcd},$$

where the mapping

$$\tau_\psi \tau_\varphi^{-1} : \tau_\varphi(\pi^{-1}(U \cap V)) \to \tau_\psi(\pi^{-1}(U \cap V))$$

is a homeomorphism of open sets into $R^n \times R^n$ (the transition homeomorphism from one set of coordinates to another). It suffices to show that $\tau_\psi \tau_\varphi^{-1} \in C^{r-1}$. Since $\tau_\varphi a = (\varphi^{-1}(x), h_1)$, $\tau_\psi a = (\psi^{-1}(x), h_2)$ and $h_2 = D_{\varphi^{-1}(x)}(\psi^{-1}\varphi)h_1$, we easily obtain that

$$\tau_\psi \tau_\varphi^{-1}(y, h) = ((\psi^{-1}\varphi)(y), D_y(\psi^{-1}\varphi)h), \tag{4}$$

whence $\tau_\psi \tau_\varphi^{-1} \in C^{r-1}$. ∎

Note that transition transformation (4) is of a special form: the coordinates of a point x are transformed by means of the diffeomorphism $\psi^{-1}\varphi$, and the vector component h of the tangent vector a by means of the linear transformation $D_{\varphi^{-1}(x)}(\psi^{-1}\varphi)$.

Thus, the structure of a smooth manifold of dimension $2n$ has been constructed on the topological space TM^n. Due to a special form of transition transformation (4), such a manifold is called the *tangent bundle* of the manifold M^n. The new term underlines the structure of the manifold TM^n which consists of fibres over

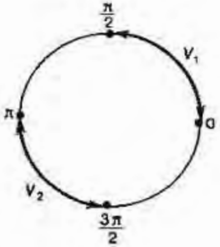


Fig. 89

each point in M^n that are tangent spaces. The smooth manifold structure determined by the atlas of charts of form (3) is called the *tangent bundle structure*.

EXAMPLE 2. Construct the structure of the tangent bundle TS^1 for the circumference $S^1 \subset R^2$. Considering S^1 as the set of points $e^{is} (s \in R^1)$, specify on S^1 the C^∞-atlas of the two charts

$$U_1 = \left\{ e^{is} : s \in \left(0, \frac{3}{2}\pi\right) \right\}, \varphi_1(s) = e^{is} : \left(0, \frac{3}{2}\pi\right) \to U_1,$$

$$U_2 = \left\{ e^{is} : s \in \left(-\pi, \frac{\pi}{2}\right) \right\}, \varphi_2(s) = e^{is} : \left(-\pi, \frac{\pi}{2}\right) \to U_2.$$

In fact,the set $U_1 \cap U_2$ consists of two connected components V_1, V_2 (Fig. 89) and the transition homeomorphism on them is of the form

$$\varphi_2^{-1}\varphi_1(s) = s : \varphi_1^{-1}(V_1) \to \varphi_2^{-1}(V_1),\ \varphi_2^{-1}\varphi_1(s) = (s - 2\pi) : \varphi_1^{-1}(V_2) \to \varphi_2^{-1}(V_2),$$

and is therefore a C^∞-diffeomorphism. The charts of the atlas on the tangent bundle are then of the form

$$\tilde{U}_1 = \pi^{-1}(U_1), \tau_{\varphi_1} : \pi^{-1}(U_1) \to \left(0, \frac{3}{2}\pi\right) \times R^1,$$

$$\tilde{U}_2 = \pi^{-1}(U_2), \tau_{\varphi_2} : \pi^{-1}(U_2) \to \left(-\pi, \frac{\pi}{2}\right) \times R^1.$$

NOTE. Since $D(\varphi_2^{-1}\varphi_1) = 1_{R^1}$, the tangent vector determined by the representative $(x, (U_1, \varphi_1), h)$ in the chart (U_1, φ_1) is of the form $(x, (U_2, \varphi_2), h)$ in the chart (U_2, φ_2). Gluing the manifold TS^1 from the direct products $\left(0, \frac{3}{2}\pi\right) \times R^1$, $\left(-\pi, \frac{\pi}{2}\right) \times R^1$ along the diffeomorphism $\tau_{\varphi_2}\tau_{\varphi_1}^{-1} = (\varphi_2^{-1}\varphi_1, 1_{R^1})$, we obviously obtain

the direct product $S^1 \times R^1$. Thus, the tangent bundle TS^1 is homeomorphic to $S^1 \times R^1$.

Exercise 6°. Let V be an open set in M^n. Show that the tangent bundle of the set V regarded as a submanifold in M^n coincides with $\pi^{-1}(V)$. Describe TV, $V \subset R^n$.

Remember that we discussed the phase space of a system in Item 1. The state of a system can be now characterized by an element from TM^n, i.e., the tangent vector a over a point x. Then x characterizes the position of the system in the configuration space, and the vector a from T_xM^n specifies the velocity of the system.

4. **The Riemannian Metric.** The tangent bundle is related to the notion which is important in geometric problems, viz., that of the Riemannian metric on a manifold. Consider a C^r-manifold M^n, $r \geq 1$, and its tangent bundle TM^n. Let a symmetric, positive-definite, bilinear function $A_x(u, v)$ that, generally speaking, depends on x be defined in each fibre of T_xM^n. We will assume that this dependence is of class C^{r-1} in the sense that in the local coordinates on the chart $(\pi^{-1}(U), \tau_\varphi)$ of the tangent bundle TM^n, the bilinear function $A_x(\tau_x^{-1}u, \tau_x^{-1}v)$ in a fixed basis for the vector space R^n has a matrix $A_{ij}(x)$ whose elements are C^{r-1}-functions on U.

The form $A_x(u, v)$ is called the *Riemannian metric* of class C^{r-1} on the manifold M^n. It is often specified in the local coordinates on the tangent bundle as a bilinear form

$$\sum_{i,j} a_{ij}(x)u_iv_j, \; x \in U,$$

where $u_1, \ldots, u_n, v_1, \ldots, v_n$ are the coordinates of the vectors u, v of the space R^n. The Riemannian metric enables us to measure the lengths of vectors and the angles between them in tangent spaces, e.g., if $v \in T_xM^n$ then the *length* $\| v \|_x$ *of a vector* v is defined by the equality $\| v \|_x^2 = A_x(v, v)$. It will be interesting to consider the question of existence of the Riemannian metric on smooth manifolds.

THEOREM 1. *On any C^r-manifold M^n, $r \geq 1$, there exists a Riemannian metric of class C^{r-1}.*

PROOF. Consider a certain atlas $\{(U_\alpha, \varphi_\alpha)\}$ on the manifold M^n. Let $\{V_\beta\}$ be a locally finite,open covering of M^n such that each V_β lies in a certain U_α (such a covering is there due to the paracompactness of M^n). For each β, we fix a certain number $\alpha = \alpha(\beta)$ and construct a C^r-partition $\{g_\beta\}$ of unity which is subordinate to the covering $\{V_\beta\}$. The idea of constructing a Riemannian metric is to construct on each V_β (as a submanifold of M^n with the tangent bundle $\pi^{-1}(V_\beta)$) its own Riemannian metric $A_x^\beta(u, v)$, and then by partitioning unity, glue the 'global' Riemannian metric from them:

$$A_x(u, v) = \sum_\beta g_\beta(x) A_x^\beta(u, v). \tag{5}$$

Exercise 7°. Verify that if $A_x^\beta(u, v)$ is a Riemannian metric on V_β (for each β), then formula (5) determines a Riemannian metric on M^n.

It remains to construct a Riemannian metric on V_β. By the previous construction, $V_\beta \subset U_{\alpha(\beta)}$, therefore $\pi^{-1}(V_\beta) \subset \pi^{-1}(U_{\alpha(\beta)})$ and $\pi^{-1}(V_\beta)$ belongs to the

chart $(\pi^{-1}(U_{\alpha(\beta)}), \tau_{\varphi_{\alpha(\beta)}})$ of the tangent bundle TM^n. Thus, we obtain the mapping

$$\tau^{\beta} = \tau_{\varphi_{\alpha(\beta)}} : \pi^{-1}(V_\beta) \to (\varphi^{-1}_{\alpha(\beta)}(V_\beta)) \times R^n. \tag{6}$$

Consider the bilinear form $B(u, v) = u_1v_1 + \ldots + u_nv_n$ with the constant matrix $(a_{ij}(x)) = (\delta_{ij})(\delta_{ij}$ being the Kronecker delta) in local coordinates (6). We now specify a Riemannian metric on V_β by the equality

$$A^\beta_x(u, v) = B(\tau^\beta_x u, \tau^\beta_x v), \quad u, v \in T_xM^n, x \in V_\beta,$$

where τ^β_x is a restriction of τ^β to the fibre T_xM^n. ■

5. **Tangential Maps.** While studying smooth mappings of surfaces (resp. curves) in analysis and its applications,they often use the linearization method which consists in replacing a surface (resp. curve) by the tangent plane (resp. straight line) in neighbourhoods of some point and its image, and replacing a mapping by its differential, i.e., by a linear mapping. This method admits generalization for the case of mappings of smooth manifolds.

Let $f : M^n \to N^m$ be a smooth mapping of class C^r, $r \geq 1$, of smooth manifolds of the same class. Let $x \in M^n$ be an arbitrary point,and $(U, \varphi)(V, \psi)$ charts in the manifolds M^n, N^m, respectively, such that $x \in U$, $f(x) \in V$; we assume also that $f(U) \subset V$. Consider the representation of the mapping f in the given coordinates

$$\psi^{-1}f\varphi : \varphi^{-1}(U) \to \psi^{-1}(U)$$

and its derivative

$$D_{\varphi^{-1}(x)}(\psi^{-1}f\varphi) : R^n \to R^m. \tag{7}$$

DEFINITION 3. Let $a \in T_xM^n$ be an arbitrary tangent vector at a point x, and $(x, (U, \varphi), h)$ its representative in the chart (U, φ). The linear mapping

$$T_x(f) : T_xM^n \to T_{f(x)}N^m,$$

under which the tangent vector a with the representative $(x, (U, \varphi), h)$ is transformed into a tangent vector b with the representative $(f(x), (V, \psi), g)$ in the chart (V, ψ), where $g = D_{\varphi^{-1}(x)}(\psi^{-1}f\varphi)h$, is called the *tangential map of f at the point $x \in M^n$*.

Thus, under a tangential map,a point x is 'carried' by a mapping f, and the vector component h, corresponding to the chosen chart, of the tangent vector is transformed by linear mapping (7).

Exercise 8°. Show that the tangential map T_xf does not depend on the choice of charts.

The verification of the following basic properties of the tangential map is left to the reader as a simple exercise:

(i) to the identity mapping $1_{M^n} : M^n \to M^n$, there corresponds the identity mapping

$$T_x(1_{M^n}) = 1_{T_xM^n} : T_xM^n \to T_xM^n,$$

(ii) the commutativity of the mapping diagram

$$\begin{tikzcd} & N^m \arrow[dd, "g"] \\ M^n \arrow[ur, "f"] \arrow[dr, "gf"'] & \\ & P^k \end{tikzcd}$$

entails the commutativity of the tangential map diagram

$$\begin{tikzcd} & T_yN^m \arrow[dd, "T_y(g)"] \\ T_xM^n \arrow[ur, "T_x(f)"] \arrow[dr, "T_x(gf)"'] & \\ & T_zP^k \end{tikzcd}$$

where $y = f(x)$, $z = g(y)$.

The collection of all tangential maps $\{T_x(f)\}_{x \in M^n}$ determines a tangent bundle mapping

$$T(f) : TM^n \to TN^m \tag{8}$$

called a *map of manifolds tangential to f.*

Using smooth structures on TM^n and TN^m, the representation of the mapping $T(f)$ in the corresponding charts can be written. Indeed, let (V, ψ) be a chart at the point $f(x)$, and (U, φ) a chart at the point x, and moreover, $f(U) \subset V$. Consider the charts $(\pi^{-1}(U), \tau_\varphi)$, $(\pi^{-1}(V), \tau_\psi)$ in the tangent bundles TM^n, TN^m, respectively. To a tangent vector $a \in \pi^{-1}(U)$, there corresponds the pair $(\varphi^{-1}(x), h)$ in the chart τ_φ; similarly, to the vector $b = T(f)a$, there corresponds the pair $(\psi^{-1}f(x)$, $D_{\varphi^{-1}(x)}(\psi^{-1}f\varphi)h)$ in the chart τ_ψ. We have the following transition transformation

$$\tau_\psi T(f)\tau_\varphi^{-1} : (y, h) \to ((\psi^{-1}f\varphi)(y), D_y(\psi^{-1}f\varphi)h)) \tag{9}$$

acting from the set $\tau_\varphi(\pi^{-1}(U)) \subset R^n \times R^n$ to the set $\tau_\psi(\pi^{-1}(V)) \subset R^m \times R^m$. It is clear that mapping (9) is of smoothness class C^{r-1}.

Thus, with each smooth mapping of manifolds of class C^r, $r \geqslant 1$, smooth mapping (8) of class C^{r-1} of their tangent bundles can be associated. For the tangential maps of tangent bundles, properties (i) and (ii) remain valid.

The definition of a regular point of a smooth mapping of manifolds (see Def. 5, Sec. 5) can be reformulated in terms of tangential map. Let $f : M^n \to N^m$ be a C^r-mapping ($r \geqslant 1$) of C^r-manifolds.

DEFINITION 4. A point $x \in M^n$ is called a *regular point* of a mapping f if rank $T_x(f) = \min(n, m)$.

Exercise 9°. Verify the equivalence of Definition 4 to Definition 5, Sec. 5.

The advantage of Definition 4 is in its being given in invariant form, i.e., in a form independent of the choice of coordinate systems.

6. Orientation of Manifolds. The notions of tangent space and tangent bundle enable us to define the concept of orientability of smooth manifolds by generalizing the definition of an orientable surface which is quite important in analysis.

Remember the notion of oriented vector space R^n. Two bases $(e_1, \ldots, e_n)$ and $(g_1, \ldots, g_n)$ in R^n are said to be of the same *orientation* if the transfer from one basis to the other is carried out by a linear mapping with a positive determinant.

Exercise 10°. Show that orientation is an equivalence relation on the set of all bases in R^n and that the number of equivalence classes equals 2.

A space R^n is said to be *oriented* if one of the equivalence classes of the bases is fixed in it.

Consider a C^r-submanifold M^n, $r \geqslant 1$, in the space R^N. A submanifold M^n is said to be *orientable* if orientations in each tangent space T_xM^n and an atlas $\{(U_\alpha, \varphi_\alpha)\}$ in M^n can be chosen such that the corresponding diffeomorphisms $\varphi_\alpha : R^n \to U_\varphi$ preserve the orientations, i.e., for any point $x \in U_\alpha$, the tangential map $T_x\varphi_\alpha^{-1} : T_xM^n \to R^n$ transforms the chosen orientation of the vector space T_xM^n into a fixed orientation of the vector space R^n.

Otherwise, the manifold is said to be *non-orientable*.

An atlas satisfying this condition is called an *orienting* atlas. It is clear that for an orienting atlas, the diffeomorphisms $\varphi_\alpha: R^n \to U_\alpha$ are compatible with each other. The precise meaning of this compatibility is expressed by the following exercise.

Exercise 11°. Show that any two charts $(U_\alpha, \varphi_\alpha)$, (U_β, φ_β) from an orienting atlas are positively compatible, i.e., possess the property of the determinant of the mapping $D_{\varphi_\alpha^{-1}(x)}(\varphi_\beta^{-1}\varphi_\alpha) : R^n \to R^n$ to be positive for any point $x \in \varphi_\alpha^{-1}(U_\alpha \cap U_\beta)$; conversely, if any two charts of an atlas are positively compatible then the atlas is orienting.

The property expressed in Exercise 11 is used in defining an orientable manifold (not necessarily embedded in R^N).

We introduce an equivalence relation on the set of orienting atlases: two orienting atlases are *equivalent* if their union is an orienting atlas.

The choice of one of the equivalence classes is called an *orientation of the manifold*.

Exercise 12°. Verify that for any manifold, the number of the equivalence classes of orienting atlases is even, and in the case of a connected manifold equals 0 or 2.

The simplest example of an orientable manifold is the space R^n. In this case, the atlas consisting of one chart (R^n, I_{R^n}) is orienting

Exercise 13°. Show that any manifold that possesses an atlas consisting of one chart is orientable.

Exercise 13° gives us another example: an open set in R^n and, therefore, any open disc D^n are orientable.

The Cartesian product of orientable manifolds is another example of an orientable manifold. We leave the proof of this fact as an exercise to the reader.

Exercises.

14°. Construct an orienting atlas on S^n.

15°. Show that the manifold $G_k(R^n)$ is orientable for an even n, $0 < k < n$.

As to non-orientable manifolds, these are, for example, a Möbius strip and the projective space RP^{n-1} for an even $n - 1 > 0$. We do not give the proof here. If $n - 1$ is odd then RP^{n-1} is orientable as it follows from Exercise 15°.

NOTE. Mind that when $n = 0$, $n = 1$, any manifold M^n is orientable.

The notion of orientation enables us to perfect the degree modulo 2 of a mapping, the notion introduced in Sec. 5. In considering a mapping of oriented manifolds, we will count the number of points in the inverse image of a regular value algebraically with a '+' or '−' sign, depending on whether or not the tangential map at this point preserves the orientation rather than counting it modulo 2. As well as in the case of the degree modulo 2, it can be shown that this number does not depend on the choice of a regular value; this is called the *degree* (oriented) *of a mapping f* and denoted by deg (f). Theorem 7, Sec. 5, is valid for the degree deg (f). In the case of smooth mappings of spheres, the degree of a mapping so defined coincides with the degree of a mapping introduced in Sec. 4, Ch. III.

7. TANGENT VECTOR AS DIFFERENTIAL OPERATOR. DIFFERENTIAL OF FUNCTION AND COTANGENT BUNDLE

1. A New Definition of a Vector. We continue the study of the tangent vector and give its definition in terms of the differentiation with respect to a vector. This enables us to give a new interpretation of the tangent bundle.

Consider the Euclidean space R^n and a C^∞-function f defined in a neighbourhood of a point $x^0 \in R^n$. Consider the vector space $R^n_{x^0}$ of all n-dimensional vectors at the point x^0. If (x^0, v) is a vector from $R^n_{x^0}$, then the derivative $\frac{d}{dt} f(x^0 + tv)\Big|_{t=0}$, where $t \geqslant 0$ is a numerical parameter, is called the *derivative of the function f with respect to the vector* v at the point x^0. (In analysis, a vector v of unit length is usually considered, and a directional derivative is spoken of.) We have the following formula in the coordinate system

$$\frac{d}{dt} f(x^0 + tv)\Big|_{t=0} = \left(\frac{\partial f_1}{\partial x^1}\right)_{x^0} v_1 + \ldots + \left(\frac{\partial f_n}{\partial x_n}\right)_{x^0} v_n = (\operatorname{grad} f(x^0), v), \quad (1)$$

where $x_1, \ldots, x_n, v_1, \ldots, v_n$ are the coordinates of the point x and vector v. Denote derivative (1) by $f_v(x^0)$. For a certain vector v and point x^0, we have obtained the correspondence $f \to f_v(x^0)$ determining a certain function (functional) $l^v_{x^0}$ which is given on smooth functions in neighbourhoods of the point x^0 and with values in R^1. It is obvious that this functional is defined on the germs $\hat{f}_{x^0}$ of smooth functions at the point x^0. Thus, we have the mapping

$$l^v_{x^0} : \mathscr{O}(x^0) \to R^1. \quad (2)$$

From the definition, the following properties of functional (2) can be deduced:

(1) $l^v_{x^0}(\hat{f}\hat{g}) = f(x^0) l^v_{x^0}(\hat{g}) + g(x^0) l^v_{x^0}(\hat{f})$ (the formula for the derivative of a product);

(2) $l_{x0}^v(\hat{f}) = 0$ if $f = \text{const}$ (the formula for the derivative of a constant);
(3) $l_{x0}^v(\alpha\hat{f} + \beta\hat{g}) = \alpha l_{x0}^v(\hat{f}) + \beta l_{x_0}^v(\hat{g})$, $\alpha, \beta \in R^1$ (the linearity).

Consider the set $\{l\}_{x0}$ of all functionals $l : \mathscr{O}(x^0) \to R^1$ that satisfy properties (1), (2) and (3). It is evident that $\{l\}_{x_0}$ is a vector space and $l_{x0}^v \in \{l\}_{x0}$. Now, if the vector v 'ranges over' the space R_{x0}^n, the mapping

$$R_{x0}^n \to \{l\}_{x0}, \; v \to l = l_{x0}^v \tag{3}$$

is given rise.

THEOREM 1. *Mapping* (3) *is an isomorphism of the vector spaces* R_{x0}^n *and* $\{l\}_{x0}$.

PROOF. The linearity of mapping (3) follows from formula (1). Mapping (3) is a monomorphism: if $l_{x0}^v = l_{x0}^w$, then $(\operatorname{grad} f(x^0), v) = (\operatorname{grad} f(x^0), w)$ for any function f which is smooth in a neighbourhood of x^0; putting $f(x) = x_i$ (the coordinate of the point x), we obtain the equalities $v_i = w_i$, $i = 1, \ldots, n$, i.e., $v = w$.
Prove that mapping (3) is epimorphic. We have

$$f(x) = f(x^0) + \sum_{i=1}^{n} A_i(x^0)(x_i - x_i^0) + \sum_{i,j=1}^{n} A_{ij}(x)(x_i - x_i^0)(x_j - x_j^0), \tag{4}$$

where

$$A_i(x^0) = \frac{\partial f}{\partial x_i}(x^0), \; i = 1, \ldots, n, \tag{5}$$

and $A_{ij}(x)$ are functions of class C^∞ (see Ex. 4, Sec. 1).

Now, let $l : \mathscr{O}(x^0) \to R^1$ be an arbitrary functional from $\{l\}_{x0}$. Using axioms (1), (2) and (3), we obtain from (4) that

$$l(\hat{f}) = \sum_{i=1}^{n} A_i(x^0) l(\hat{x}_i - x_i^0) = \sum_{i=1}^{n} A_i(x^0) l(\hat{x}_i),$$

where $l(\tilde{x}_i)$ is the value of l on the germ of the function x_i, i.e., on the coordinate of x.

Using (5), we obtain, finally, that

$$l(\hat{f}) = \left(\frac{\partial f}{\partial x_1}\right)\bigg|_{x^0} v_1 + \ldots + \left(\frac{\partial f}{\partial x_n}\right)\bigg|_{x^0} v_n = l_{x0}^v(f), \tag{6}$$

where $v_1 = l(\hat{x}_1), \ldots, v_n = l(\hat{x}_n)$. ■

Due to isomorphism (3), the vector space R_{x0}^n can be identified with the n-dimensional vector space $\{l\}_{x0}$ of all functionals satisfying axioms (1), (2) and (3). By means of the coordinate system in R^n and using equality (6), each functional l_{x0} can be associated with the differential operator

$$\sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}\bigg|_{x^0} \tag{7}$$

acting on smooth functions according to the formula

$$\left(\sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}\bigg|_{x^0}\right) f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)\bigg|_{x^0} v_i.$$

Exercise 1°. Verify that the set of all differential operators (7) forms a vector space, and the indicated correspondence specifies an isomorphism with the vector space $\{l\}_{x^0}$.

Thus, we have another isomorphism, i.e., that of the vector space $R^n_{x^0}$ with the vector space of differential operators (7). Under this isomorphism, to the basis vector $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (where 1 is in the i-th place), there corresponds the differential operator $\dfrac{\partial}{\partial x_i}\bigg|_{x^0}$.

2. Tangent Bundles. The interpretation of the space of vectors at a point x^0 given in Item 1 leads to the corresponding generalization of this notion for smooth manifolds.

Let M^n be a manifold of class C^∞, and x^0 a point from M^n. Consider the algebra $\mathcal{O}(x^0)$ of germs of smooth functions at a point x^0 (see Sec. 4) and the functionals

$$l_{x^0} : \mathcal{O}(x^0) \to R^1. \tag{8}$$

Exercise 2°. Let (U, φ) be a chart at a point x^0 of a manifold M^n. Verify that the functional l_{x^0} determined by the equality

$$l_{x^0}(\hat{f}) = l^v_{\varphi^{-1}(x^0)}(\hat{f}\varphi), \hat{f} \in \mathcal{O}(x^0)$$

for any vector $v \in R^n$ specifies functional (8) satisfying axioms (1), (2) and (3).

DEFINITION 1. The set of all functionals (8) satisfying properties (1), (2) and (3) is called the *tangent space* $T_{x^0}M^n$ to the manifold M^n at the point x^0.

The tangent space $T_{x^0}M^n$ is a vector space with the natural algebraic operations. An individual element l_{x^0} from $T_{x^0}M^n$ is called a *tangent vector* to the manifold M^n at the point x^0.

The correspondence $l^v_{\varphi^{-1}(x^0)} \to l_{x^0}$ (see Ex. 2) happens to be an isomorphism of the spaces $R^n_{\varphi^{-1}(x^0)}$ and $T_{x^0}M^n$. In fact, the linearity of the mapping is obvious, and the inverse mapping is given by the formula

$$l_{\varphi^{-1}(x^0)}(\hat{g}) = l_{x^0}(\hat{g}\varphi^{-1}), \hat{g} \in \mathcal{O}(\varphi^{-1}(x^0)),$$

where $\mathcal{O}(\varphi^{-1}(x^0))$, $\mathcal{O}(x^0)$ are the algebras of the germs at the points $\varphi^{-1}(x^0) \in R^n$, $x^0 \in M^n$, respectively. It is convenient to assume, henceforward, that the functional l_{x^0} is given not only on the germs $\hat{g} \in \mathcal{O}(x^0)$ but also on the functions g which are defined in a neighbourhood of the point x^0 (we set $l_{x^0}(g) = l_{x^0}(\hat{g})$), and write $l_{x^0}(g)$ instead of $l_{x^0}(\hat{g})$.

Let $\Phi : M^n \to N^m$ be a smooth mapping of manifolds and let $x^0 \in M^n$, $y^0 = \Phi(x^0) \in N^m$. The mapping Φ induces the mapping $\hat{\Phi} : \mathcal{O}(y^0) \to \mathcal{O}(x^0)$ between the algebras of the germs according to the rule $\hat{g} \in \mathcal{O}(y^0)$, $\hat{g} \to \hat{f}$, $f = g\Phi$. This enables us to define the tangential map $T_{x^0}\Phi : T_{x^0}M^n \to T_{y^0}N^m$ by the rule $T_{x^0}(\Phi)l_{x^0} = l_{y^0}$, where $l_{y^0} = l_{x^0}\hat{\Phi}$.

The action of the mappings Φ, $\hat{\Phi}$ is shown in the following diagrams

$$\begin{tikzcd} M^n \arrow[r, "\Phi"] \arrow[dr, dashed, "f=g\Phi"'] & N^m \arrow[d, "g"] \\ & R^1 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}(x^0) \arrow[dr, "l_{x^0}"'] & & \mathcal{O}(y^0) \arrow[ll, "\hat{\Phi}"'] \arrow[dl, dashed, "l_{y^0}=l_{x^0}\hat{\Phi}"] \\ & R^1 & \end{tikzcd}$$

Exercise 3°. Verify that $l_{x0}\hat{\Phi}$ is the tangent vector at the point y^0 of the manifold N^m and that $T_{x0}(\Phi)$ is a linear mapping.

The tangential map $T_{x0}(\Phi)$ is often denoted by $(\Phi_*)_{x0}$ (or $d_{x0}\Phi$).

Exercises.

4°. Show that $[(1_{M^n})_*]_{x0} = 1_{T_{x0}M^n}$. If $\Phi : M^n \to N^m$, $\Psi : N^m \to P^k$ are smooth mappings of manifolds then $[(\Psi\Phi)_*]_{x0} = (\Psi_*)_{\Phi(x^0)}(\Phi_*)_{x0}$.

5°. Prove that if Φ is a diffeomorphism, then $(\Phi_*)_{x0}$ is an isomorphism of vector spaces (and therefore $m = n$).

We now pass on to the construction of the tangent bundle. Just like in Sec. 6, put $TM^n = \bigcup_{x \in M^n} T_xM^n$ (disjoint union). The problem is to determine the structure of the tangent bundle on TM^n. We specify the projection $\pi : TM^n \to M^n$ by associating an element $l_x \in T_xM^n$ with a point $x \in M^n$. Let (U, φ) be some chart at the point x. We construct the chart on the tangent space corresponding to the chart (U, φ)

$$\tau_\varphi : \pi^{-1}(U) \to R^n \times R^n. \tag{9}$$

Let $l_x \in T_xM^n$. Then the tangent vector $\tilde{l}_{\varphi^{-1}(x)}$ is defined on the algebra $\mathcal{O}(\varphi^{-1}(x))$ according to the rule

$$\tilde{l}_{\varphi^{-1}(x)}(g) = l_x(g\varphi^{-1}),\ g \in \mathcal{O}(\varphi^{-1}(x)).$$

In view of the isomorphism of the space of differential operators and the tangent space (see Ex. 1), we have

$$\tilde{l}_{\varphi^{-1}(x)} = v_1 \frac{\partial}{\partial x_1} + \ldots + v_n \frac{\partial}{\partial x_n}, \tag{10}$$

where the differential operators $\dfrac{\partial}{\partial x_i}$ act at the point $\varphi^{-1}(x) = (x_1, \ldots, x_n)$ and $v = (v_1, \ldots, v_n)$ is a vector determined uniquely. Mapping (9) is given by the correspondence which is linear on each fibre $\pi^{-1}(x)$:

$$l_x \to (x_1, \ldots, x_n; v_1, \ldots, v_n). \tag{11}$$

The bijectivity of this mapping is evident; as in Sec. 6, we define the topology on TM^n by the condition for the continuity of mappings τ_φ for all charts of a certain atlas on the manifold M^n.

Show that mappings (9) and (11) determine the structure of a tangent bundle. If (V, ψ) is another chart at the point x then the tangent vector $l_{\psi^{-1}(x)}$ is determined similarly:

$$l_{\psi^{-1}(x)} = w_1 \frac{\partial}{\partial y_1} + \ldots + w_n \frac{\partial}{\partial y_n}, \tag{12}$$

where $w = (w_1, \ldots, w_n)$ is a vector at the point $\psi^{-1}(x) = (y_1, \ldots, y_n)$, and the mapping τ_ψ acts according to the rule $l_x \to (y_1, \ldots, y_n; w_1, \ldots, w_n)$.

Let us calculate $\tau_\psi \tau_\varphi^{-1}$. We have the following mapping of class C^∞:

$$\begin{aligned}(y_1, \ldots, y_n) &= \psi^{-1}\varphi(x_1, \ldots, x_n)\\ &= ((\psi^{-1}\varphi)_1(x_1, \ldots, x_n), \ldots, (\psi^{-1}\varphi)_n(x_1, \ldots, x_n).\end{aligned} \tag{13}$$

Expressing $w_1, \ldots, w_n$ in terms of $v_1, \ldots, v_n$ and setting $g \in \mathcal{O}(\psi^{-1}(x))$, we derive the following equality from (12)

$$l_{\psi^{-1}(x)}(g) = w_1 \frac{\partial g}{\partial y_1} + \ldots + w_n \frac{\partial g}{\partial y_n},$$

but

$$l_{\psi^{-1}(x)}(g) = l_x(g\psi^{-1}) = l_x(g\psi^{-1}\varphi\varphi^{-1}) = l_{\psi^{-1}(x)}(g\psi^{-1}\varphi).$$

Now, using formula (10),

$$l_{\psi^{-1}(x)}(g) = v_1 \frac{\partial(g\psi^{-1}\varphi)}{\partial x_1} + \ldots + v_n \frac{\partial(g\psi^{-1}\varphi)}{\partial x_n},$$

and comparing two expressions for $l_{\psi^{-1}(x)}(g)$,

$$\sum_{i=1}^{n} w_i \frac{\partial g}{\partial y_i} = \sum_{j=1}^{n} v_j \frac{\partial(g\psi^{-1}\varphi)}{\partial x_j}.$$

Since the germ $g \in \mathcal{O}(\psi^{-1}(x))$ is arbitrary, we may put $g(y_1, \ldots, y_n) = y_i$. Then, from (13), we obtain:

$$(g\psi^{-1}\varphi)(x_1, \ldots, x_n) = (\psi^{-1}\varphi)_i(x_1, \ldots, x_n),$$

and from the previous equality,

$$w_i = \sum_{j=1}^{n} v_j \frac{\partial(\psi^{-1}\varphi)_i}{\partial x_j}(x_1, \ldots, x_n), \quad i = 1, \ldots, n. \tag{14}$$

Transforming the coordinates as in (13), the vector component of the tangent vector is transformed by linear transformation (14) with the Jacobian matrix $\left(\frac{\partial(\psi^{-1}\varphi)}{\partial x}\right)$, i.e., by the linear transformation $D_{\varphi^{-1}(x)}(\psi^{-1}\varphi)$. Transformations (13) and (14) smoothly depend on the point $\varphi^{-1}(x)$ and thus determine a transfor-

mation $\tau_\psi \tau_\varphi^{-1}$ of class C^∞. The special form of this coordinate transformation means that the atlas $\{(\pi^{-1}(U), \tau_\varphi)\}$ on TM determines the smooth tangent bundle structure.

Exercise 6°. Consider the Euclidean space R^n with the structure given by the atlas consisting of one chart $(R^n, 1_{R^n})$. Verify that $T_x R^n$ is isomorphic to R^n_x.

We may now consider TR^n to be the set of all pairs (x, v), where $x \in R^n$ and $v \in R^n_x$. We may also assume that

$$TR^n = \left\{ \left(x_1, \ldots, x_n; v_1 \frac{\partial}{\partial x_1} + \ldots + v_n \frac{\partial}{\partial x_n} \right) \right\},$$

where $x_1, \ldots, x_n$ are the coordinates of x, and $v_1, \ldots, v_n$ are the coordinates of v; the mapping $\tau_{1_{R^n}}$ determines the only chart of the corresponding atlas on the tangent bundle TR^n:

$$\left(x_1, \ldots, x_n; v_1 \frac{\partial}{\partial x_1} + \ldots + v_n \frac{\partial}{\partial x_n} \right) \to (x_1, \ldots, x_n, v_1, \ldots, v_n). \quad (15)$$

Thereby, the direct product structure is introduced on TR^n which is said to be the *trivial* tangent bundle.

Exercise 7°. Show that mapping (9) decomposes into the product $\tau_\varphi = \tau_{1_{R^n}}(\varphi^{-1})_*$ acting according to the rule

$$l_x \xrightarrow{(\varphi^{-1})^*} \left(x_1, \ldots, x_n, v_1 \frac{\partial}{\partial x_1} + \ldots + v_n \frac{\partial}{\partial x_n} \right)$$
$$\xrightarrow{\tau_{1_{R^n}}} (x_1, \ldots, x_n, v_1, \ldots, v_n).$$

It is often convenient to work not with the coordinate representation of the vector l_x, but with its image $(\varphi^{-1})_* l_x$, the vector component of the latter being $v_1 \dfrac{\partial}{\partial x_1} + \ldots + v_n \dfrac{\partial}{\partial x_n}$. On changing a chart (changing the coordinates), its coordinates $v_1, \ldots, v_n$ in the basis $\left\{ \dfrac{\partial}{\partial x_i} \right\}_{i=1}^n$ are transformed by formulae (14) into the coordinates $w_1, \ldots, w_n$ in the basis $\left\{ \dfrac{\partial}{\partial y_i} \right\}_{i=1}^n$.

3. Tangential Maps. Let $\Phi : M^n \to N^m$ be a smooth mapping of manifolds. For each $x \in M^n$, we have a linear mapping $(\Phi_*)_x : T_x M^n \to T_y N^m$, where $y = \Phi(x)$. Thereby, the mapping $\Phi_* : TM^n \to TN^m$ is defined. Let us verify that Φ_* is a smooth mapping of tangent bundles.

Let (U, φ) be a chart at the point x, (V, ψ) a chart at the point y, $l_x \in T_x M^n$, and $l_y = (\Phi_*)_x l_x$. Due to (11), we have

$$l_x \xrightarrow{\tau_\varphi} (x_1, \ldots, x_n; v_1, \ldots, v_n), \; l_y \xrightarrow{\tau_\psi} (y_1, \ldots, y_m; w_1, \ldots, w_m), \quad (16)$$

and it is now required to find the transformation

$$\tau_\psi(\Phi_*)\tau_\varphi^{-1} : (x_1, \ldots, x_n; v_1, \ldots, v_n) \to (y_1, \ldots, y_m; w_1, \ldots, w_m). \quad (17)$$

Since $(\psi^{-1}\Phi\varphi)(x_1, \ldots, x_n) = (y_1, \ldots, y_m)$, it remains to find the relation between $\{v_i\}$ and $\{w_i\}$. If $g \in \mathscr{O}(\psi^{-1}y))$ then the following equalities are valid

$$\begin{aligned} \tilde{l}_{\psi^{-1}(y)}(g) &= l_y(g\psi^{-1}) = ((\Phi_*)_x l_x)(g\psi^{-1}) = l_x(g\psi^{-1}\Phi) = l_x(g\psi^{-1}\Phi\varphi\varphi^{-1}) \\ &= \tilde{l}_{\varphi^{-1}(x)}(g\psi^{-1}\Phi\varphi). \end{aligned}$$

But taking into account that the first and the last functionals are equal to

$$\sum_{j=1}^{m} w_j \frac{\partial g}{\partial y_j} \quad (18)$$

and

$$\sum_{j=1}^{n} v_j \frac{\partial(g\psi^{-1}\Phi\varphi)}{\partial x_j}, \quad (19)$$

respectively, we obtain, just like in deducing (14), by equalizing (18) to (19) and putting $g \equiv y_i$, that

$$w_i = \sum_{j=1}^{n} v_j \frac{\partial(\psi^{-1}\Phi\varphi)_i}{\partial x_j}, \; i = 1, \ldots, m. \quad (20)$$

Hence, mapping (17) is of class C^∞. Formula (20) affirms the fact established earlier that the vector component of a tangent vector is transformed by means of the linear transformation $D_{\varphi^{-1}(x)}(\psi^{-1}\Phi\varphi)$.

4. The Differential of a Function and a Cotangent Bundle. Consider the action of a vector $l_{x^0} \in T_{x^0}M^n$ on the function f, $f \in \mathscr{O}(x^0)$. If the function f is fixed then there arises a linear functional on the space $T_{x^0}M^n : l_{x^0} \to l_{x^0}(f)$. This functional is denoted by the symbol $(\mathrm{d}f)_{x^0}$ and called the *differential of the function* f at the point x^0.

By definition,

$$(df)_{x^0} l_{x^0} = l_{x^0}(f).$$

Thus, $(\mathrm{d}f)_{x^0}$ belongs to $(T_{x^0}M^n)^*$, the space conjugate to the space $T_{x^0}M^n$ having the natural vector structure.

Let (U, φ) be a chart at the point x^0, and $\{x_i(x)\}_{i=1}^n$ the local coordinates of a point $x \in U$. Below, we identify tangent vectors with their corresponding differential operators. Let $\left\{\dfrac{\partial}{\partial x_i}\bigg|_{x^0}\right\}_{i=1}^{n}$ be a basis for $T_{x^0}M^n$, and $(\mathrm{d}x_i)_{x^0}$ the differential of the function $x_i(x)$, then

$$(\mathrm{d}x_i)_{x^0}\left(\frac{\partial}{\partial x_j}\bigg|_{x^0}\right) = \left(\frac{\partial}{\partial x_j} x_i\right)_{x_0} = \delta_{ij} \quad (21)$$

($\delta_{ij} = 0$ when $i \neq j$, $\delta_{ii} = 1$). Therefore $\{(\mathrm{d}x_i)_{x^0}\}_{i=1}^n$ is the basis dual of $\left\{\frac{\partial}{\partial x_i}\bigg|_{x^0}\right\}_{i=1}^n$ in $(T_{x^0}M^n)^*$. Hence it follows also that $(T_{x^0}M^n)^*$ consists of all possible linear combinations $[a_1(\mathrm{d}x_1)_{x^0} + \ldots + a_n(\mathrm{d}x_n)_{x^0}]$ with real coefficients. For an arbitrary function f, $f \in \mathcal{O}(x^0)$ and vector $l_{x^0} = v_1 \frac{\partial}{\partial x_1}\bigg|_{x^0} + \ldots + v_n \frac{\partial}{\partial x_n}\bigg|_{x^0}$, we have the decomposition:

$$(\mathrm{d}f)_{x^0}l_{x^0} = l_{x^0}(f) = v_1 \frac{\partial f}{\partial x_1}(x^0) + \ldots + v_n \frac{\partial f}{\partial x_n}(x^0).$$

By means of (21), we obtain

$$(\mathrm{d}x_i)_{x^0}l_{x^0} = (\mathrm{d}x_i)_{x^0}\left(\sum_{j=1}^{n} v_j \frac{\partial}{\partial x_j}\bigg|_{x^0}\right) = v_i$$

and, substituting in the previous equality, we find:

$$(\mathrm{d}f)_{x^0}l_{x^0} = \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x^0)(\mathrm{d}x_i)_{x^0}\right) l_{x^0}.$$

Due to the arbitrariness of $l_{x^0} \in T_{x^0}M^n$, we have

$$(\mathrm{d}f)_{x^0} = \frac{\partial f}{\partial x_1}(x^0)(\mathrm{d}x_1)_{x^0} + \ldots + \frac{\partial f}{\partial x_n}(x^0)(\mathrm{d}x_n)_{x^0}.$$

Replacing x^0 by an arbitrary point $x \in U$, the latter formula can be rewritten in a more convenient way:

$$(\mathrm{d}f)_x = \frac{\partial f}{\partial x_1}(x)\mathrm{d}x_1 + \ldots + \frac{\partial f}{\partial x_n}(x)\mathrm{d}x_n. \tag{22}$$

Here $\{\mathrm{d}x_i\}_{i=1}^n$ are the basis differentials at the point x. Formula (22) justifies the name of a 'differential' for $(\mathrm{d}f)_x$.

Let us deduce from (22) the relation between the differentials of the coordinates of various local coordinate systems at the point x. Let (V, ψ) be a chart that defines the coordinates $\{y_i(x)\}_{i=1}^n$, $x \in V$. If $x \in U \cap V$ then the coordinates $\{x_i(x)\}_{i=1}^n$ and $\{y_i(x)\}_{i=1}^n$ are related by the transformation $\psi^{-1}\varphi$ (see (13)). From (22), we have the equalities

$$(\mathrm{d}y_i)_x = \frac{\partial y_i}{\partial x_1}(x)\mathrm{d}x_1 + \ldots + \frac{\partial y_i}{\partial x_n}(x)\mathrm{d}x_n,$$

but

$$\frac{\partial}{\partial x_j} y_i(x) = \frac{\partial}{\partial x_j} (\psi^{-1}\varphi)_i(x_1, \ldots, x_n),$$

therefore

$$(\mathrm{d}y_i)_x = \sum_{j=1}^{n} \frac{\partial (\psi^{-1}\varphi)_i}{\partial x_j} dx_j, i = 1, \ldots, n \tag{23}$$

Thus, in transition from one system of local coordinates to another, the differentials of the coordinates considered as functions of a point on a manifold are transformed by formulae (23), i.e., by the linear transformation given by the Jacobian matrix $\left(\frac{\partial(\psi^{-1}\varphi)_i}{\partial x_j}\right)$.

Consider the disjoint union $T^*M = \bigcup_{x \in M^n} (T_xM^n)^*$. We construct the structure of the vector bundle on T^*M^n. The natural projection $p . T^*M^n \to M^n$ is defined. Let (U, φ) be a chart on M^n, $\{x_i(x)\}$ the local coordinates of the point $x \in U$. We define on T^*M^n a chart

$$\sigma_\varphi : p^{-1}(U) \to R^n \times R^n, \tag{24}$$

by specifying the mapping σ_φ according to the rule

$$\sum_{i=1}^{n} a_i dx_i \to (x_1, \ldots, x_n; a_1, \ldots, a_n), \tag{25}$$

where $\sum_{i=1}^{n} a_i dx_i$ is an element from the fibre $p^{-1}(x) = (T_xM^n)^*$.

Let us show that $\{(p^{-1}(U), \sigma_\varphi)\}$ is an atlas on the C^∞-structure if $\{(U, \varphi)\}$ is the atlas on the manifold M^n. Let (V, ψ) be another chart at the point x, determining the local coordinates $\{y_i(x)\}_{i=1}^n$, and

$$\sigma_\psi : p^{-1}(V) \to R^n \times R^n \tag{26}$$

another chart on T^*M^n. It is clear that $p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V)$. Consequently, if $x \in U \cap V$ then to the element $\sum_{i=1}^{n} a_i dx_i$, the coordinates in chart (26) may be assigned:

$$\sum_{i=1}^{n} a_i dx_i \to (y_1, \ldots, y_n; b_1, \ldots, b_n). \tag{27}$$

We conclude from (25) and (27) that the elements in $(T_xM^n)^*$ coincide:

$$\sum_{i=1}^{n} a_i (dx_i)_x = \sum_{i=1}^{n} b_i (dy_i)_x. \tag{28}$$

It is not complicated to derive the relation between $[a_i]$ and $[b_j]$ from (28). In fact, substituting the expression $(dy_i)_x$ from (23) in (28), and then equalizing the coefficients of the same differentials dx_i on both sides of the equality, we obtain:

$$\begin{pmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix} = \left(\frac{\partial(\psi^{-1}\varphi)_i}{\partial x_j}\right)^{*-1} \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} .$$

where $*$ denotes the operation of transposing the matrix. Hence, the vector component of an element of the fibre $p^{-1}(x)$ is altered in the coordinate representation on changing the coordinates by a rule which is different from that for the vector component of the tangent vector, viz., it is transformed by means of the matrix $\left(\frac{\partial(\psi^{-1}\varphi)}{\partial x}\right)^{*-1}$, whereas the vector component of the tangent vector is transformed by the Jacobian matrix $\left(\frac{\partial(\psi^{-1}\varphi)}{\partial x}\right)$. The quantities which change by this rule on changing the coordinate system are called *covectors*. The elements of the set $(T_xM^n)^*$ are called *covectors at the point x*.

It is clear now that on constructing charts (25) for all charts of a certain atlas on M^n, we will transform the set of all covectors, i.e., T^*M^n, into a smooth manifold; this manifold is called a *cotangent bundle*.

8. VECTOR FIELDS ON SMOOTH MANIFOLDS

The concepts treated in this section are important both for a great number of mathematical disciplines (such as differential equations, dynamic systems, topology of manifolds) and for applications to mechanics and physics. Here, these relations will be outlined in the most elementary form. For the simplicity of enunciations, we shall consider all the objects to be of class C^∞, calling them smooth, just like we did in Sec. 7.

1. The Tangent Vector to a Smooth Path.

Let M^n be a smooth manifold. Recall that a path in M^n is a continuous mapping $\chi : (a, b) \to M^n$ of an interval of the number line into the topological space M^n. Since (a, b) is a smooth submanifold in R^1, smooth mappings χ can be considered while referring to the path as *smooth*.

Let χ be a smooth path in M^n, and $\chi(t)$ a point of this path, $t \in (a, b)$.

DEFINITION 1. The *tangent vector to a path* χ at a point $\chi(t)$ is the tangent vector $l_{\chi(t)}$ to the manifold M^n at the point $\chi(t)$ determined by the equality

$$l_{\chi(t)}(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(\chi(t))|_t, f \in \mathcal{O}(\chi(t)). \tag{1}$$

Exercise 1°. Verify that the right-hand side in (1) determines a tangent vector to the manifold M^n.

The tangent vector to the path $\chi(t)$ is usually denoted by $\chi'(t)$. Let us find the coordinate representation of the vector $\chi'(t)$. Let (U, φ) be a chart at the point $\chi(t)$. Consequently, if $g \in \mathcal{O}(\varphi^{-1}(\chi(t)))$, then

$$l_{\chi(t)}(g\varphi^{-1}) = \frac{\mathrm{d}}{\mathrm{d}t}(g\varphi^{-1}\chi(t))|_t = \sum_{i=1}^{n} \frac{\mathrm{d}x_i(t)}{\mathrm{d}t}\frac{\partial g}{\partial x_i}(x_i(t), \ldots, x_n(t)),$$

where $\varphi^{-1}\chi(t) = (x_1(t), \ldots, x_n(t))$ is the corresponding path in R^n. Hence, we derive the coordinates of the vector $\chi'(t)$ in the chart (U, φ):

$$\tau_\varphi l_{\chi(t)} = \left(x_1(t), \ldots, x_n(t); \frac{\mathrm{d}x_1(t)}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}x_n(t)}{\mathrm{d}t}\right). \tag{2}$$

Here $x'(t) = (x_1'(t), \ldots, x_n'(t))$ is the vector component of the tangent vector.

If $A_\chi(u, v)$ is the Riemannian metric on M^n, then the length $\|\chi\ (t)\|_{\chi(t)}$ of the tangent vector to the path and the length of the portion of the path when $t_1 \leqslant t \leqslant t_2$ are determined:

$$S_{t_1}^{t_2} = \int_{t_1}^{t_2} \sqrt{A_{\chi(t)}(\chi'(t), \chi'(t))}\ dt = \int_{t_1}^{t_2} \|\chi'(t)\|_{\chi\ (t)}\mathrm{d}t. \tag{3}$$

In local coordinates, formula (3) is the following

$$S_{t_1}^{t_2} = \int_{t_1}^{t_2} \sqrt{g_{ij}(x(t))\frac{\mathrm{d}x_i}{\mathrm{d}t}\frac{\mathrm{d}x_j}{\mathrm{d}t}}\,\mathrm{d}t = \int_{t_1}^{t_2} \sqrt{g_{ij}(x(t))\mathrm{d}x_i\,\mathrm{d}x_j},$$

where $g_{ij}(x)$ is the corresponding matrix of the bilinear form.

2. The Dynamical Group of a Physical System and Its Infinitesimal Generator. The notions of smooth path and its tangent vector find natural application to the mathematical investigation of physical systems.

We will speak of the set of all possible states of a physical system in some process and call it the phase space of the system while assuming that it is a smooth manifold M^n. Then $x \in M^n$ denotes a possible state of the system, and the correspondence 'a point x — a state' is bijective. The state of the system varies with time in accordance with a law of physics, and therefore the point x corresponding to this state changes its position F with time. We will assume the process to be determinate, which means that the state of the system is determined uniquely, in future and in the past, by its present state. Such processes are described by the dynamical group of a physical system defined as follows: if $x \in M^n$ is a point marking the state of the system at the present moment (when $t = 0$), then to the state of the system at a moment t, there corresponds the point $\chi = \chi(t, x)$, $\chi(t, x) \in M^n$, $\chi(0, x) = x$. Thus, the point x describes the path $\chi = \chi(t, x)$, $-\infty < t < +\infty$ called the phase trajectory (orbit)

of the point x. For any $t \in (-\infty, +\infty)$, the transformation $U_t : M^n \to M^n$ is defined by the rule $x \to \chi(t, x)$. Due to the determinacy principle, we have:

$$U_{t_1 + t_2}(x) = U_{t_1}(U_{t_2}x), t_1, t_2 \in (-\infty, +\infty).$$

Hence the family of the transformations $\{U_t\}$ is a group with the inverse element $(U_t)^{-1} = U_{-t}$ and the unit element $U_0 = 1_{M^n}$.

This group is called the *dynamical group* of the physical system. We will assume that the mapping $R^1 \times M^n \to M^n : (t, x) \to U_t(x)$ is smooth; in this case, the group of the diffeomorphisms $\{U_t\}$ is said to *depend smoothly on t*. From the point of view of physics, to know the dynamical group means to possess a complete description of the behaviour of the system with time. Such a description is not always possible.

The laws of physics are generally formulated much simpler in 'infinitesimal form', which means the following: consider an orbit $\chi(t) = \chi(t, x)$ and its tangent vector $\chi'(0)$ (at a point x). For any point $x \in M^n$, we put $X(x) = \chi'(0)$. The collection of tangent vectors $\{X(x)\}$ is called the *vector field* on the manifold M^n; this field is also called the *infinitesimal generator of the dymanical group*. A physical law is usually expressed by describing an infinitesimal generator. But then the problem of constructing (describing) the dynamical group is given rise.

Below, we shall study the notion of vector field in greater detail.

3. **Smooth Vector Field.** A *vector field* on a manifold M^n is a mapping

$$X : M^n \to TM^n \tag{4}$$

such that $X(x) \in T_xM^n$ for each $x \in M^n$. A vector field is said to be *smooth* (of class C^∞) if mapping (4) is smooth (of class C^∞). In local coordinates, the vector field is of the form

$$\left(x_1, \ldots, x_n; X_1(x_1, \ldots, x_n)\frac{\partial}{\partial x_1} + \ldots + X_n(x_1, \ldots, x_n)\frac{\partial}{\partial x_n}\right). \tag{5}$$

Exercise 2°. Show that the smoothness of a vector field is equivalent to the smoothness of the functions $X_i(x_1, \ldots, x_n)$, $i = 1, \ldots, n$.

Let U_t be the group of diffeomorphisms of a manifold M^n, depending smoothly on t, and let $\chi(t, x) = U_t(x)$ be the orbit of a point x.

DEFINITION 2. A vector field $X(x)$ is called the *infinitesimal generator of the group* U_t if for any orbit $\chi(t) = \chi(t, x)$, we have:

$$\chi'(0) = X(x). \tag{6}$$

Exercises.

3°. Show that equality (6) is equivalent to the equality

$$x'(t) = X(\chi(t)), t \in (-\infty, \infty). \tag{7}$$

Hint: Use the equality $\chi(t) = U_t(x)$ and a group property $(U_sU_t(x) = U_{s+t}(x))$.

4°. Show that the infinitesimal generator $X(x)$ is a smooth vector field.

Consider the problem of seeking a group U_t for a given smooth vector field $X(x)$. We will look for the orbit $\chi(t) = \chi(t, x_0)$ using condition (7). In local coordinates, we have a system of differential equations

$$\frac{dx_i}{dt} = X_i(x_1(t), \ldots, x_n(t)), i = 1, \ldots, n, \tag{8}$$

where $(x_1(t), \ldots, x_n(t))$ is the coordinate specification of the path χ, and $X_1, \ldots, X_n$ are the coordinates of the vector component of the tangent vector $X(x)$ (see (2) and (5)). The functions $X_i(x_1, \ldots, x_n)$ depend smoothly on $x_1, \ldots, x_n$. To find the orbit (more precisely, that portion of it which lies in the chart), it is necessary to find a solution of system of differential equations (8) such that satisfies the condition $x_1(0) = x_1^0, \ldots, x_n(0) = x_n^0$, where $(x_1^0, \ldots, x_n^0)$ are the coordinates of the point x^0. Using the theory of ordinary differential equations, a unique solution $\chi(t) = \chi(t, x)$ may be found for a sufficiently small interval $-\varepsilon < t < \varepsilon$ and x from a certain neighbourhood of the point x^0, $\chi(t, x)$ depending smoothly on t, x. But to construct the group U_t, it is necessary to extend the solution $\chi(t, x)$ to the whole axis $-\infty < t < \infty$ for any $x \in M^n$. This cannot be always done (an example being the equation $y' = y^2$ on R^1). However, if M^n is compact then the required extension exists, which can be easily verified by the methods of the theory of ordinary differential equations. In this theory, the following theorem is proved.

THEOREM 1. *If M^n is a compact,smooth manifold and X a smooth vector field then the latter is the infinitesimal generator of a one-parameter group of diffeomorphisms which depends smoothly on the parameter.*

Note that orbits are often called *integral curves* of the vector field.

EXAMPLE 1. Let $M^{2n} = TQ^n$ (i.e., the case considered in mechanics, where Q^n is the configuration space which we assume hereafter to be a smooth manifold). Let the local coordinates in Q^n be $(q_1, \ldots, q_n)$, and those in $TQ^n (q_1, \ldots, q_n; v_1, \ldots, v_n)$.

The vector field on TQ^n of the form (the vector component)

$$L_{q,v} = v_1 \frac{\partial}{\partial q_1} + \ldots + v_n \frac{\partial}{\partial q_n} + \alpha_1(q, v) \frac{\partial}{\partial v_1} + \ldots + \alpha_n(q, v) \frac{\partial}{\partial v_n}$$

with smooth functions $\alpha_1(q, v), \ldots, \alpha_n(q, v)$ is said to be *special*. The integral curves of this field are described by the system of differential equations

$$\frac{dq_i}{dt} = v_i, \frac{dv_i}{dt} = \alpha_i(q_1, \ldots, q_n; v_1, \ldots, v_n), i = 1, \ldots, n,$$

which is equivalent to the system of the second order

$$\frac{d^2 q_i}{dt^2} = \alpha_i \left(q_1, \ldots, q_n, \frac{dq_1}{dt}, \ldots, \frac{dq_n}{dt} \right), i = 1, \ldots, n. \tag{9}$$

Classical mechanics operates with equations of form (9).

Exercise 5°. Find $\pi_* L_{q,v}$, where $\pi : TQ^n \to Q^n$ is the projection.

4. **The Lie Algebras of Vector Fields.** Let $C^\infty(M^n)$ be the set of all smooth mappings $f : M^n \to R^1$. A smooth vector field X on M^n determines the mapping $C^\infty(M^n) \to C^\infty(M^n)$ by the rule $f \to X(f)$, where $X(f)(x) = X(x)(f)$ for any point x from M^n. If (in local coordinates)

$$X(x) = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i}, \quad \text{then} \quad X(f)(x) = \sum_{i=1}^{n} a_i(x) \frac{\partial f}{\partial x_i}.$$

It is obviously a linear mapping of the vector space $C^\infty(M^n)$.

If X, Y are two vector fields then their product, the *commutator* $[X, Y]$, may be defined by the formula

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

Exercise 6°. Verify that $[X, Y]$ is a vector field and calculate it in local coordinates; show that $[X, Y] = -[Y, X]$.

It is evident that the set of all vector fields on M^n forms a vector space (over the field R^1) under the natural operations $X + Y$ and $\alpha \cdot X$. The commutator $[X, Y]$ depends linearly on the factors X, Y. Thus, the set of all vector fields on M^n is an algebra called a *Lie algebra*.

Exercise 7°. Prove the Jacobi identity

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$$

for any three vector fields X, Y, Z on M^n.

The set of all vector fields on a smooth manifold is not only a vector space but also possesses the structure of the module over the ring of smooth functions on M^n. In fact, for $f \in C^\infty(M^n)$, the product $f \cdot X$ is defined which is a vector field $(f \cdot X)(x) = f(x) \cdot X(x)$; it is clear that this is a smooth field depending both on f and X linearly.

The study of the structure of the Lie algebras of vector fields is one of the main trends in modern topology.

5. **Covector Fields.** A smooth mapping $A : M^n \to T^*M^n$ under which $A(x)$ belongs to $T_x^*M^n$, i.e., the fibre over the point x, is called a *smooth covector field* on M^n.

In local coordinates, the field A is given in the form

$$A(x) = (x_1, \ldots, x_n; a_1(x)\mathrm{d}x_1 + \ldots + a_n(x)\mathrm{d}x_n),$$

where $a_i(x)$ are smooth functions in the coordinates $(x_1, \ldots, x_n)$ of the point x.

If $f \in C^\infty(M^n)$ then $A(x) = (\mathrm{d}f)_x$ is a covector field on M^n. It is smooth, since in local coordinates the covector component $(\mathrm{d}f)_x$ is of the form

$$\frac{\partial f}{\partial x_1} dx_1 + \ldots + \frac{\partial f}{\partial x_n} \mathrm{d}x_n,$$

and the coordinates $\dfrac{\partial f}{\partial x_i}$ of this covector are smooth functions.

EXAMPLE 2. Let $M^{2n} = T^*Q^n$, where Q^n is the configuration space of a mechanical system. Let $(q_1, \dots, q_n, p_1, \dots, p_n)$ be the local coordinates on T^*Q^n, where $(p_1, \dots, p_n)$ are the coordinates of the covector at the point $(q_1, \dots, q_n) \in Q^n$. If $H : T^*Q^n \to R^1 \to$ is a smooth function then

$$dH = \frac{\partial H}{\partial q_1} dq_1 + \dots + \frac{\partial H}{\partial q_n} dq_n + \frac{\partial H}{\partial p_1} dp_1 + \dots + \frac{\partial H}{\partial p_n} dp_n$$

is a covector field on the manifold T^*Q^n. Let us form a vector field on the same manifold, viz.,

$$L = -\frac{\partial H}{\partial p_1}\frac{\partial}{\partial q_1} - \dots - \frac{\partial H}{\partial p_n}\frac{\partial}{\partial q_n} + \frac{\partial H}{\partial q_1}\frac{\partial}{\partial p_1} + \dots + \frac{\partial H}{\partial q_n}\frac{\partial}{\partial p_n}.$$

The integral curves $(q_1 t), \dots, q_n(t); p_1(t), \dots, p_n(t))$ of the field L satisfy the system of differential equations

$$\frac{dq_i}{dt} = -\frac{\partial H}{\partial p_i}, \frac{dp_i}{dt} = \frac{\partial H}{\partial q_i}, i = 1, \dots, n. \tag{10}$$

In mechanics, they choose the function H to be the sum of kinetic and potential energies, and call system (10) the *equations of motion in Hamiltonian form.*

9. FIBRE BUNDLES AND COVERINGS

1. Preliminary Examples. Many problems naturally give rise to spaces which are arranged, locally, as the direct products of spaces. At present, they have been studied quite thoroughly. However, we will only touch upon the very first concepts.

Consider examples of spaces having, locally, the direct product structure.

Let M^n be a smooth manifold, TM^n a tangent bundle and $\pi : TM^n \to M^n$ the projection of the tangent bundle onto the manifold. It is clear that for any point $x \in M^n$, the fibre $\pi^{-1}(x)$ is homeomorphic to the space R^n and, moreover, for the coordinate neighbourhood U of the point x, we have the homeomorphism $\pi^{-1}(U) = U \times R^n$ (see Sec. 6). However, generally speaking, one cannot assert that there exists a homeomorphism $TM^n = M^n \times R^n$ as, for example, in the case when $M^n = S^1$.

The tangent bundle is arranged locally (which is, certainly, a corollary to its definition) as the direct product $U \times R^n$. That the sphere S^3 has a similar structure is more surprising and related to the properties of complex numbers. We shall construct an example of a mapping of S^3 to S^2 for which the inverse image of any point is homeomorphic to the circumference. Consider S^3 as a sphere in C^2, i.e.,

$$S^3 = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\},$$

and the sphere S^2 as the extended complex plane (z-sphere). The formula $\pi(z_1, z_2) = z_1/z_2$ defines the mapping $\pi : S^3 \to S^2$. For $\lambda = e^{i\alpha}$, we have $\pi(\lambda z_1, \lambda z_2) = \pi(z_1, z_2)$, therefore $\pi^{-1}(z) = S^1$ for any $z \in S^2$. Remember that the

z-sphere S^2 possesses the C^∞-manifold structure with the local coordinate z in the region $U_1 = S^2 \setminus \infty$ and $1/z$ in the region $U_2 = S^2 \setminus 0$ (see Sec. 2).

Consider the direct products $U_1 \times S^1$, $U_2 \times S^1$. The sets $\pi^{-1}(U_1)$, $\pi^{-1}(U_2)$ turn out to be homeomorphic to $U_1 \times S^1$ and $U_2 \times S^1$, respectively. To show it, we define the mapping $\varphi : U_1 \to S^3$ by the formula

$$\varphi(z) = \left(\frac{z}{\sqrt{1 + |z|^2}}, \frac{1}{\sqrt{1 + |z|^2}}\right).$$

It is evident that $\pi\varphi = 1|_{U_1}$ and the set $\pi^{-1}(z)$, $z \in U_1$ consists of points of the form $\lambda\varphi(z)$, where $\lambda = e^{i\alpha}$. We define the mapping $\tilde{\varphi} : U_1 \times S^1 \to S^3$ by the formula

$$\tilde{\varphi}(z, \lambda) = \left(\frac{\lambda z}{\sqrt{1 + |z|^2}}, \frac{\lambda}{\sqrt{1 + |z|^2}}\right), z \in U_1, \lambda \in S^1.$$

It is clear that $\pi^{-1}(U_1) = \tilde{\varphi}(U_1 \times S^1)$ and the diagram

$$\begin{tikzcd} U_1 \times S^1 \arrow[rr, "\tilde{\varphi}"] \arrow[dr, "pr_1"'] & & \pi^{-1}(U_1) \arrow[dl, "\pi"] \\ & U_1 & \end{tikzcd}$$

where pr_1 is the projection of the direct product onto the first factor, is commutative. Similarly, we define the mapping $\tilde{\psi} : U_2 \times S^1 \to S^3$ by the formula

$$\tilde{\psi}(1/z, \lambda) = \left(\frac{\lambda \cdot 1}{\sqrt{1 + |1/z|^2}}, \frac{\lambda \cdot (1/z)}{\sqrt{1 + |1/z|^2}}\right), 1/z \in U_2, \lambda \in S^1,$$

with $\pi^{-1}(U_2) = \tilde{\psi}(U_2 \times S^1)$. It is clear that the diagram

$$\begin{tikzcd} U_2 \times S^1 \arrow[rr, "\tilde{\psi}"] \arrow[dr, "pr_1"'] & & \pi^{-1}(U_2) \arrow[dl, "\pi"] \\ & U_2 & \end{tikzcd}$$

is commutative.

Thus, the mapping π is arranged locally (over the coordinate neighbourhoods of S^2) as the projection of the direct product. However, the sphere S^3 is not homeomorphic to the direct product $S^2 \times S^1$ (the fundamental groups of these spaces being non-isomorphic).

The described mapping is called the *Hopf mapping*; it is remarkable in many respects. Thus, for example, the Hopf mapping determines the generator of the group $\pi_3(S^2) \cong Z$. Note that for any two points u, $v \in S^2$, the circumferences $\pi^{-1}(u)$ and $\pi^{-1}(v)$ are linked in S^3 (Fig. 90).

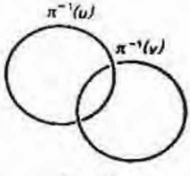


Fig. 90

2. The Definition of a Fibre Bundle.

The examples considered in Item 1 naturally lead to the following definition.

DEFINITION 1. A *locally trivial fibre space* is a quadruple (E, B, F, p), where E, B, F are spaces, p a surjective mapping of E onto B, and, moreover, for any $x \in B$, there exists a neighbourhood U of the point x and a homeomorphism $\varphi_U : p^{-1}(U) \to U \times F$ such that the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi_U"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "pr"] \\ & U & \end{tikzcd}$$

where pr is the natural projection, is commutative.

It follows from the definition that for any point x from U, the inverse image $p^{-1}(x)$ is homeomorphic to the space F. It is called a fibre over the point x.

The spaces E, B, F are called the *total space*, the *base space* and the *fibre*, while the mapping p is called the *projection*, respectively. The neighbourhoods U involved in Definition 1 are called *coordinate neighbourhoods* and the homeomorphisms φ_U *coordinate* or *rectifying homeomorphisms*.

Though a wider, than locally trivial fibre spaces, class of *fibre bundles* may be considered in topology, by a fibre bundle we will mean hereafter a locally trivial fibre space.

A fibre bundle is said to be *trivial* if there exists a homeomorphism $\varphi_B : E \to B \times F$ such that the diagram

$$\begin{tikzcd} E \arrow[rr, "\varphi_B"] \arrow[dr, "p"'] & & B \times F \arrow[dl, "pr"] \\ & B & \end{tikzcd}$$

is commutative.

Thus, the tangent bundle TM^n can be considered as the total space of a locally trivial fibre space with the base space M^n, projection $p = \pi$, i.e., projection of the tangent space onto the manifold M^n, and fibre R^n. As neighbourhoods $U \subset M^n$, the coordinate neighbourhoods of the manifold M^n may be taken.

The mapping considered above is the projection of the Hopf bundle, i.e., of the locally trivial fibre space whose total space is S^3, base space S^2 and fibre S^1.

We now list some other examples of locally trivial fibre spaces.

EXAMPLES.

1. THE MÖBIUS STRIP M (i.e., the factor space of the direct product $[0, 1] \times [-1, 1]$ relative to the equivalence $(0, y) \sim (1, -y)$) is a total space with the base space S^1 (the 'median') and fibre $[-1, 1]$.

The projection $pr : [0, 1] \times [-1, 1] \to [0, 1]$ acting according to the rule $pr(x, y) = x$ induces the residue class mapping $p : M \to S^1$, i.e., the projection of this fibre bundle.

2. THE DIRECT PRODUCT $X \times Y$ OF TOPOLOGICAL SPACES X, Y forms a total space with the natural projection $pr : X \times Y \to X$, fibre Y and the base space X.

3. THE SPHERE S^n is a total space with the base space RP^n, a fibre consisting of two points (discrete set) and the projection associating a point $x \in S^n$ with its equivalence class $\{x, -x\} \in RP^n$ (see Sec. 5, Ch. II).

THE SPHERE S^{2n+1} is a total space with the base space CP^n, fibre S^1 and projection associating a point $x \in S^{2n+1} \subset C^{n+1}$ with its equivalence class in CP^n (see Sec. 5, Ch. II).

Exercises.

1°. Show that the tangent bundle of a manifold M^n is trivial if and only if there exist n (continuous) vector fields on M^n such that they are linearly independent at each point $x \in M^n$.

2°. Show that the locally trivial fibre space over a line-segment is trivial.

A mapping $s : B \to E$ satisfying the condition $ps = 1_B$ is called a *cross-section* of the fibre bundle (E, B, F, p).

Exercises.

3°. Show that the existence of a cross-section is a necessary condition for the triviality of a fibre bundle.

4°. Do there exist cross-sections of the Hopf bundle? (Use the equality $\pi_2(S^3) = 0$.)

5°. Give an example of a nontrivial fibre bundle which possesses a cross-section.

Let us establish a certain relation between mappings to a total space and to its base space.

DEFINITION 2. A mapping $\Psi : X \to E$ is called a *lift of a mapping* $\Phi : X \to B$ if for any point $x \in X$, the equality $p\Psi(x) = \Phi(x)$ holds. The mapping Ψ is also said to *cover* the mapping Φ.

The introduced relation is characterized by the commutativity of the following diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\Psi"] \arrow[r, "\Phi"'] & B \end{tikzcd}$$

Exercise 6°. Show that if a fibre bundle possesses a cross-section, then for any mapping to the base space, there exists a lift of this mapping.

We now give a necessary condition for the existence of a lift of a mapping in terms of functors of homotopy groups (see Sec. 3, Ch. III).

THEOREM 1. *Let (E, B, F, p) be a locally trivial fibre space whose total space and base space are path-connected, and X a path-connected topological space. For the mapping $\Phi : X \to B$ to have a lift Ψ satisfying the condition $\Psi(x_0) = e_0$, where $x_0 \in X$, $e_0 \in E$, $p(e_0) = b_0 = \Phi(x_0)$, it is necessary that*

$$\Phi_n(\pi_n(X, x_0)) \subseteq p_n(\pi_n(E, e_0)) \tag{3}$$

for all $n \geqslant 1$.

PROOF. If such a lift Ψ exists, then diagram (2) is commutative. Using functors of homotopy groups, we obtain the commutative diagrams (for all $n \geqslant 1$)

$$\begin{tikzcd}
& \pi_n(E, e_0) \arrow[d, "p_n"] \\
\pi_n(X, x_0) \arrow[ur, "\Psi_n"] \arrow[r, "\Phi_n"] & \pi_n(B, b_0)
\end{tikzcd}$$

from which the required inclusions follow easily. ■

Locally trivial fibre spaces possess the following important property.

THE COVERING HOMOTOPY PROPERTY. Let (E, B, F, p) be a locally trivial fibre space whose base space is Hausdorff and paracompact. Let X be an arbitrary topological space, $\Phi : X \times I \to B$ a homotopy, and $f : X \to E$ a lift of the mapping $\Phi|_{X \times 0}$. Then there exists a unique lift $\Psi : X \times I \to E$ of the homotopy Φ satisfying the condition $\Psi|_{X \times 0} = f$.

This statement will be proved for a special case in Item 4.

3. Vector Bundles.

Let (E, B, F, p) be a locally trivial fibre space. Assume that U and V are two coordinate neighbourhoods of a point $x \in B$. Homeomorphisms $g_V^U(x)$ of the space F may be given by the formula

$$g_V^U(x)h = \varphi_V \varphi_U^{-1}(x, h), x \in U \cap V, h \in F; g_U^U(x) = 1_F.$$

If W is a third neighbourhood of the point x, then the following equalities are valid:

$$g_W^U(x) = g_W^V(x) g_V^U(x).$$

Thus, for any point $x \in U \cap V$, the homeomorphism $g_V^U(x)$ is defined, i.e., the mapping $g_V^U : U \cap V \to H(F)$ of the set $U \cap V$ to the group $H(F)$ of homeomorphisms of the space F is given; the mappings g_V^U are called the *coordinate transformations*. If F is locally compact and the topology on $H(F)$ is induced by the embedding of $H(F)$ into the space $C(F, F)$ with the compact-open topology, then the coordinate transformations are easily seen to be continuous (see Ex. 11, Sec. 1, Ch. III).

DEFINITION 3. A *vector bundle* is a locally trivial fibre space (E, B, F, p) whose fibre F is a finite-dimensional vector space and whose coordinate transformations g_V^U are continuous mappings to the group of invertible linear transformations of the

space F (for fixed U and V, $g_V^U(x)$ is a family of invertible linear operators which is continuously dependent on $x \in U \cap V$).

Exercise 7°. Show that the tangent bundle TM^n is a vector bundle.

DEFINITION 4. A *morphism of a locally trivial fibre space* (E, B, F, p) to a locally trivial fibre space (E', B', F', p') is a pair of continuous mappings $H : E \to E'$, $h : B \to B'$ such that $hp = p'H$.

The last equality implies that the diagram

$$\begin{array}{ccc} E & \xrightarrow{H} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[h]{} & B' \end{array}$$

is commutative (the fibre is transformed into the fibre).

This definition transforms the collection of locally trivial fibre spaces into a category.

DEFINITION 5. Let (E, B, F, p) and (E', B', F', p') be vector bundles whose fibres F and F' are vector spaces over the same field, and (H, h) a morphism of (E, B, F, p) to (E', B', F', p'). The morphism (H, h) is called a *morphism of vector bundles* if for any point $x \in B$, the superposition

$$F \xrightarrow{\varphi_x^{-1}} p^{-1}(x) \xrightarrow{H} (p')^{-1}(h(x)) \xrightarrow{\varphi'_h(x)} F'$$

is a linear mapping, where φ_x, $(\varphi'_{h(x)})$ are the homeomorphisms of the fibre $p^{-1}(x)$, $((p')^{-1}h(x))$ and vector space F, (F'), which are given rise in the commutative diagram of Definition 1.

Exercises.

8°. Verify that vector bundles and their morphisms form a category.

9°. Verify that by associating a manifold with a tangent bundle, a smooth mapping of manifolds with a tangential map of fibre spaces, we define a covariant functor from the category of smooth manifolds to the category of vector bundles (over the field R).

4. Coverings. We now dwell on one special class of locally trivial fibre spaces examining it in greater detail.

Consider the circumference $S^1 = \{z \in C^1 : |z| = 1\}$. We define a mapping $p : R^1 \to S^1$ by the formula $p(t) = e^{2\pi it}$. Since $p(t_1) = p(t_2)$ if and only if $t_1 - t_2 = k$, $k \in Z$, the inverse image $p^{-1}(z)$ of any point $z \in S^1$ is homeomorphic to the set of integers Z with the discrete topology. For any point $z \in S^1$, the mapping p homeomorphically maps each connected component of the set $p^{-1}(S^1 \setminus z) = R^1 \setminus p^{-1}(z)$ onto $S^1 \setminus z$. The many-valued mapping $p^{-1} : S^1 \setminus z \to R^1 \setminus p^{-1}(z)$, i.e., $(1/2\pi i) \ln u$, possesses a countable number of one-valued branches. Denote one of them by φ.

Now, we define the homeomorphism $\tilde{\varphi} : (S^1 \setminus z) \times Z \to R^1 \setminus p^{-1}(z)$ by the formula $\tilde{\varphi}(u, k) = \varphi(u + k)$. We obtain the commutative diagram

$$\begin{tikzcd}
(S^1\setminus z)\times Z \arrow[rr, "\tilde{\varphi}"] \arrow[dr, "pr_1"'] & & p^{-1}(S^1\setminus z) \arrow[dl, "p"] \\
& S^1\setminus z &
\end{tikzcd}$$

The family of sets $\{S^1\setminus z\}_{z\in S^1}$ can be taken to be the coordinate neighbourhood system so that the quadruple (R^1, S^1, Z, p) is a locally trivial fibre space whose fibre Z is discrete. Such fibre bundles are often encountered in problems of analysis.

DEFINITION 6. A locally trivial fibre space (E, B, F, p) is called a *covering* if the total space E and the base space B of the fibre space are path-connected and the fibre F is a space with the discrete topology.

Speaking of coverings, instead of the quadruple (E, B, F, p) and where it does not introduce ambiguity, they often consider the mapping $p : E \to B$ which is a surjection. The fibre $p^{-1}(x)$ over each point of the covering is homeomorphic to the space F with the discrete topology and hence is a discrete space itself.

NOTE. In the definition of a covering (and also of a locally trivial fibre space with a path-connected base space), the requirements for the homeomorphism φ_U may be weakened by assuming that φ_U is a homeomorphism onto $U \times F_U$, where F_U is a space with the discrete topology depending on the coordinate neighbourhood U. With such a definition, it is evident that $p^{-1}(x) \sim F_U$ (bijection and hence a homeomorphism) for any $x \in U$. But it happens so that $F_U \sim F_V$ (bijection and homeomorphism) for any coordinate neighbourhoods U, V, and if we put $F = p^{-1}(x_*)$, where x_* is a certain point from B, then Definition 6 (or 1) will follow (see Note after the proof of Lemma 1).

EXAMPLES.

4. The fibre bundle of the sphere S^n over the projective space RP^n is a covering whose fibre consists of two points.
5. The mapping $p : S^1 \to S^1$ $(C\setminus 0 \to C\setminus 0)$ given by the correspondence $z \mapsto z^n$ is a covering map whose fibre consists of n points.

A covering whose fibre consists of n points is called an *n-sheeted covering*.

Note that for the coordinate neighbourhood U of the covering (E, B, F, p), the inverse image $p^{-1}(U)$ is homeomorphic to the product $U \times F$ consisting of disjoint 'sheets', viz., the open sets $U \times \alpha$, $\alpha \in F$, and therefore, consists of disjoint 'sheets', viz., the open sets $W_\alpha = \varphi_U^{-1}(U \times \alpha)$, which are homeomorphic to U itself; the homeomorphisms are $p_\alpha : W_\alpha \to U$, i.e., the restrictions of p to W_α, which follows from the relation $p = pr\varphi_U$ expressing the commutativity of Diagram 1.

Thus, the projection of the covering map $p : E \to B$ is a *local homeomorphism* with the discrete inverse image $p^{-1}(x)$ which is homeomorphic to the fibre F over each point $x \in B$.

However, the converse does not hold: the covering of the space B by the coordinate neighbourhoods U cannot be constructed for every local homeomorphism $p : E \to B$ (for example, for the mapping $p : (a, b) \to S^1$ of a number interval onto the circumference, given by the formula $p(t) = (\cos t, \sin t)$).

The points from the fibre $p^{-1}(x)$ are said to lie 'over' the point x, and the sheets W_α 'over' U; the above property of the projection of the covering map $p : E \to B$ enables us to *'lift'* the subsets $A \subset U$ to the sheet W_α by considering the inverse images $p_\alpha^{-1}(A)$ and also 'to lift' mappings, paths, homotopies in X. In accordance with Definition 2, the path $f : I \to E$ is called a *lift* of the path $g : I \to B$ (which is *covering* the path g) if $pf = g$.

LEMMA 1. Let $p : E \to B$ be a covering map. Then the following statements are true: (i) any path γ in B starting at a point $b_0 \in B$ possesses a unique covering path $\widetilde{\gamma}$ in E which starts at any point $e_0 \in p^{-1}(b_0)$; (ii) if $\gamma = \gamma_1 \cdot \gamma_2$ is the product of paths γ_1 and γ_2 in B then the covering path is $\widetilde{\gamma} = \widetilde{\gamma}_1 \cdot \widetilde{\gamma}_2$, where $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ cover γ_1 and γ_2, respectively; (iii) if $\gamma = \gamma_1^{-1}$ is the path inverse of γ_1, then $\widetilde{\gamma} = \widetilde{\gamma}_1^{-1}$.

PROOF. Let a path γ be given by a mapping $\gamma : I \to B, I = [0, 1], g(0) = b_0$. Each point of the path $\gamma(t)$ belongs to some coordinate neighbourhood U_t, and there is a connected neighbourhood (i.e., an interval) Ω_t of a point $t \in I$ such that $\gamma(\Omega_t) \subset U_t$. We pick a finite covering $\{\Omega_{s_j}\}_1^k$ out of an open covering $\{\Omega_t\}$ of the line-segment I. Let δ be the Lebesgue number of the covering $\{\Omega_{s_j}\}_1^k$. Let us break the line-segment I into segments $\Delta_i = [t_{i-1}, t_i]$ of length less than δ by division points $t_i, i = 0, \ldots, N$, $t_0 = 0$, $t_N = 1$. Then $\gamma(\Delta_i)$ lies in a certain coordinate neighbourhood U_i, $i = 1, \ldots, N$. Therefore, each portion γ_i of the path γ given by the mapping $\gamma_i : \Delta_i \to B$ admits a lift $\widetilde{\gamma}_i$ to the sheet W_{α_i} given by the mapping $f_i = p_{\alpha_i}^{-1}\gamma_i : \Delta_i \to W_{\alpha_i}$, where $p_{\alpha_i} : W_{\alpha_i} \to U_i$ is a homeomorphism onto the coordinate neighbourhood U_i. We choose a sheet containing the point $e_0 \in p^{-1}(b_0)$ as W_{α_1} and lift the portion of the path γ_1. Then $\widetilde{\gamma}_1$ starts at the point e_0. When $W_{\alpha_{i-1}}$ has already been chosen and the portion of the path γ_{i-1} lifted, we choose as W_{α_i} the sheet containing the terminal point $f_{i-1}(t_{i-1})$ of the portion of the path $\widetilde{\gamma}_{i-1}$, which lies over the point $f_{i-1}(t_{i-1})$. Then the portion of the path $\widetilde{\gamma}_i$ originates at the point $f_{i-1}(t_{i-1})$. We lift thus all the portions $\gamma_i, i = 1, \ldots, N$, of the path γ. Since the mappings $f_i : \Delta_i \to E, i = 1, \ldots, N$ are compatible on the common ends of the adjacent intervals Δ_i, they can be combined into the mapping $f : I \to E$, $f(t) = f_i(t)$ for $t \in \Delta_i$. The mapping f is just what determines the path $\widetilde{\gamma}$ that covers the path γ. The uniqueness of the covering path follows from p being a local homeomorphism. Thereby, statement (i) is proved. Statements (ii) and (iii) are obvious. ■

NOTE. The structure of the coordinate neighbourhoods $U_i, i = 1, \ldots, N$, which cover the path $\gamma : I \to B$ allows us to prove the bijectivity of the fibres F_U over the coordinate neighbourhoods if the homeomorphism φ_U acts from $p^{-1}(U)$ to $U \times F_U$ (see Definition 6 of a covering and subsequent reasoning). This is obvious if $U \cap V \neq \varnothing$, since for $x \in U \cap V$, we have $p^{-1}(x) \sim F_U$ and $p^{-1}(x) \sim F_V$. If $U \cap V = \varnothing$ then having chosen points $x \in U, y \in V$, we join them with a path $\gamma : I \to B$ using the fact that B is path-connected. For the indicated covering U_i, the fibres F_{U_i} are homeomorphic (in the discrete topology), whence $p^{-1}(x) \sim p^{-1}(y)$ and $F_U \sim F_V$. ■

Using the lemma proved, it is now easy to prove the covering homotopy theorem mentioned in Item 2 for the case of coverings.

THEOREM 2. (THE COVERING HOMOTOPY PROPERTY.) *Let (E, B, F, p) be a covering, X a topological space, $f : X \to E$ a mapping, and $\Phi : X \times I \to B$ a homotopy such that $pf = \Phi|_{X \times 0}$. Then there exists a unique lift Ψ of the homotopy Φ, i.e., a homotopy $\Psi : X \times I \to E$ such that $\Psi|_{X \times 0} = f$ and $p\Psi = \Phi$.*

PROOF. For any $x \in X$, the homotopy $\Phi : X \times I \to B$ determines a path $g_x : I \to B$, where $g_x(t) = \Phi(x, t)$, $t \in I$. The point $f(x)$ lies over the point $g_x(0) = \Phi(x, 0)$. According to Lemma 1, the path g_x can be lifted to E, $\tilde{g}_x : I \to E$ under the condition that $\tilde{g}_x(0) = f(x)$ in a unique way. Put $\Psi(x, t) = \tilde{g}_x(t)$. Thus, we have a mapping $\Psi : X \times I \to E$. It covers the mapping Φ since

$$p\Psi(x, t) = p\tilde{g}_x(t) = g_x(t) = \Phi(x, t), (x, t) \in X \times I.$$

When $t = 0$, it is evident that $\Psi(x, 0) = f(x)$, i.e., $\Psi|_{X \times 0} = f$.
The proof of the theorem is completed with the following exercise.
Exercise 10°. Prove that the mapping $\Psi : X \times I \to E$ is continuous.

The lemma on lifting a path to a covering space and the covering homotopy theorem enable us to study the relation between the fundamental groups of the base space B and the covering space E. At first, we formulate the immediate geometric corollaries to the indicated propositions.

LEMMA 2. *Let $p : E \to B$ be a covering map $e_0 \in E$, $b_0 = p(e_0) \in B$ base points. Then the following statements hold:*
(i) *If α is a closed path in E with the origin at the point e_0 and homotopic* to a constant path, then $\beta = p\alpha$ is a closed path with the origin at the point b_0 and also homotopic to a constant path.*
(ii) *If α is a path in E with the origin at the point e_0 covering a closed path β, then the homotopy of the path β is lifted to the fixed-end of the path α homotopy.*
(iii) *If α is a path in E with the origin at the point e_0 covering a closed path β homotopic to a constant path, then α is also closed and homotopic to a constant path.*

PROOF. Statement (i) is obvious due to the continuity of the mapping p. Let us prove statement (ii). Let $\beta : I \to B$, $\beta(0) = \beta(1) = b_0$ be a closed path in B and $f_t : I \to B$, $0 \leqslant t \leqslant 1$, $f_0 = \beta$ its homotopy. Denote the lift of the homotopy $f_t : p\Psi_t = f_t$, $0 \leqslant t \leqslant 1$, by $\Psi_t 1 : I \to E$, $0 \leqslant t \leqslant 1$, $\Psi_0 = \alpha$. Since the ends of the path f_t are fixed, i.e., $f_t(0) = f_t(1) = b_0$ for all t, the ends $\Psi_t(0)$, $\Psi_t(1)$ of the path Ψ_t belong to $p^{-1}(b_0)$ for all t and depend on t continuously. Since the topology of the fibre $p^{-1}(b_0)$ is discrete, the ends of the path $\Psi_t(0)$, $\Psi_t(1)$ are constant, i.e., the homotopy of the path α takes place with the ends $\alpha(0) = e_0$, $\alpha(1)$ fixed. Finally, we prove statement (iii). Let f_t be a homotopy of the path β, and Ψ_t a homotopy of the path α covering f_t. From the data, f_1 is a constant mapping and $f_1(I) = b_0$, whence $\Psi_1(I) \subseteq p^{-1}(b_0)$, i.e., Ψ_1 is a mapping into the fibre over b_0. Since the image $\Psi_1(I)$ of the line-segment I is a connected set and the topology of the fibre is discrete, $\Psi_1(I) = e_0'$; in particular, $\Psi_1(0) = \Psi_1(1) = e_0'$. From statement (ii), Ψ_t is a fixed-end homotopy, i.e., $\Psi_t(1) = \alpha(1)$, $\Psi_t(0) = \alpha(0)$, $0 \leqslant t \leqslant 1$. Therefore,

* As in Ch. III, here we consider fixed-ends homotopies of paths.

$\alpha(1) = \alpha(0) = e_0$, $\Psi_1(I) = e_0$, i.e., the path α is closed and homotopic to a constant path. ■

Let us study the relation between the fundamental groups of the total space of a covering and the base space.

The projection $p : E \to B$ induces a homomorphism of the fundamental groups $\pi_1(E)$ and $\pi_1(B)$ (see Sec. 3, Ch. III). The action of this homomorphism is described by the following theorem.

THEOREM 3. *The homomorphism $p_* : \pi_1(E) \to \pi_1(B)$ of the fundamental groups which is induced by the projection of a covering is a monomorphism.*

PROOF. Let $x_0 \in E$, $b_0 \in B$ be base points, and $p(x_0) = b_0$; let $\pi_1(E, x_0)$, $\pi_1(B, b_0)$ be fundamental groups, and $p_* : \pi_1(E, x_0) \to \pi_1(B, b_0)$ a homomorphism induced by the projection $p : E \to B$. Consider the inverse image $p_*^{-1}(e)$ of the unit element of the group $\pi_1(B, b_0)$. It suffices to show that $p_*^{-1}(e) = e'$, where e' is the unit element of the group $\pi_1(E, x_0)$. If $[\alpha] \in p_*^{-1}(e)$ then α covers the path $\beta = p\alpha$ which is homotopic to a constant path in B. According to statement (iii), α is also homotopic to a constant path (in E), and therefore $[\alpha] = e'$. ■

Thus, it follows from Theorem 3 that the group $\pi_1(E)$ is isomorphic to a subgroup of the group $\pi_1(B)$ (viz., the subgroup $G = p_*(\pi_1(E))$. Consider the cosets (e.g., right) of the subgroup G of the group $\pi_1(B)$. The following important theorem is valid.

THEOREM 4. *For any covering map $p : E \to B$, the fibre $p^{-1}(b_0)$ is in bijective correspondence with the family of cosets of the subgroup G of the group $\pi_1(B)$.*

PROOF. Let us associate the homotopy class $[\beta] \in \pi_1(B, b_0)$ with a point $e_\beta \in p^{-1}(b_0)$ by the following rule: we lift the path β to the path α in E with the origin at the point e_0 (lemma on lifting a path) and put $e_\beta = \alpha(1)$; by Lemma 2 (statement (ii)), the end of the path α does not depend on the choice of a representative $\beta \in [\beta]$, therefore, the mapping $\pi_1(B, b_0) \to p^{-1}(b_0)$, $[\beta] \to e_\beta$ is defined. If $[\beta_1]$, $[\beta_2]$ belong to the same coset, then $[\beta_1] \cdot [\beta_2]^{-1} \in p_*(\pi_1(E, e_0))$; consequently, the loop $\beta_1 \cdot \beta_2^{-1}$ with the origin at b_0 is homotopic to a certain loop $p\alpha$, where α is a loop in E with the origin at e_0. Denote the lift of the loop $\beta_1 \cdot \beta_2^{-1}$ with the origin at e_0 by α' and note that the loops α' and α are homotopic with the ends fixed (statement (ii) of Lemma 2), therefore α' is a closed loop covering the loop $\beta_1 \cdot \beta_2^{-1}$ But by Lemma 1 (statements (ii) and (iii)), $\alpha' = \tilde{\beta}_1 \cdot \tilde{\beta}_2^{-1}$. The closedness of the path $\tilde{\beta}_1 \cdot \tilde{\beta}_2^{-1}$ implies the coincidence of the origin of the path $\tilde{\beta}_1$ with that of the path $\tilde{\beta}_2$, and also of their ends. Therefore, $e_{\beta_1} = e_{\beta_2}$. Thus, the mapping $[\beta] \to e_\beta$ is constant on the cosets. Meanwhile, to different cosets there correspond different images. In fact, if we assume the contrary, then there are $[\beta_1]$ and $[\beta_2]$ from different cosets, but $e_{\beta_1} = e_{\beta_2}$, which means that the ends (and the origins) of the lifts of $\tilde{\beta}_1$, $\tilde{\beta}_2$ coincide. Therefore, $\tilde{\beta}_1 \cdot \tilde{\beta}_2^{-1}$ is a loop in E with the origin at the point e_0, $p(\tilde{\beta}_1 \cdot \tilde{\beta}_2^{-1}) = \beta_1 \cdot \beta_2^{-1}$ is a loop (with the origin at the point b_0), whence the homotopy class $[\beta_1 \cdot \beta_2^{-1}] = [\beta_1] \cdot [\beta_2]^{-1}$ of this loop belongs to $p_*(\pi_1(E, e_0))$, i.e., $[\beta_1]$, $[\beta_2]$ are from the same coset, which is contrary to the assumption. Finally, it remains to show that any point $\tilde{e} \in p^{-1}(b_0)$ is the image e_β for a certain $[\beta]$. Consider a path α joining in E the point e_0 to the point $\tilde{e}_0$ (using the condition that E is

path-connected) and put $\beta = p\alpha$; β is a closed path in B with the origin at the point b_0, the path α is its lift. Therefore $e_\beta = \bar{e}_0$. ■

COROLLARY. If the total space of the covering map $p: E \to B$ is 1-connected, i.e., $\pi_1(E) = 0$, then the fibre F and the fundamental group $\pi_1(B)$ are in bijective correspondence.

PROOF. Let us fix $x_0 \in E$, $p(x_0) = b_0$ and consider $\pi_1(E, x_0) = e'$, $\pi_1(B, b_0)$. We have $p_*(\pi_1(E, x_0)) = e$. Consequently, the family of cosets coincides with the set $\pi_1(B, b_0)$. Thus, $\pi_1(B, b_0) \sim p^{-1}(b_0) \sim F$ (the equivalence is a bijection).

DEFINITION 7. A covering (E, B, F, p) is said to be *universal* if the space E is 1-connected, i.e., $\pi_1(E) = 0$. The space E is then called a *universal covering space.*

To know the universal coverings of certain spaces is useful in calculating the fundamental group $\pi_1(B)$.

EXAMPLES.

6. The covering map $p: R^1 \to S^1, p(t) = e^{2\pi it}, F = Z$. We know already (Sec. 3, Ch. III) that $\pi_1(R^1) = 0$. Therefore, $\pi_1(S^1) \sim Z$ (cf. Sec. 4, Ch. III).
7. The covering map $p: S^n \to RP^n$ with the fibre $F = Z_2$, $n \geqslant 2$. We have $\pi_1(S^n) = 0$, $n \geqslant 2$, whence $\pi_1(RP^n) \sim Z_2$.

However, the obtained results are not complete. Having established the bijection of the group $\pi_1(B)$ with some group, we cannot be sure that the bijection preserves the group operations, i.e., is a homomorphism of groups. Let us strengthen Theorem 4 and the corollary in this aspect by assuming that in the total space E, the action compatible with the structure of a covering map of a certain group G is given.

We will consider the group G acting (from the left) on the space E and identify, for brevity, an element $g \in G$ with the corresponding homeomorphism $h_g: E \to E$ (see Sec. 5, Ch. II).

DEFINITION 8. A group G is said to *act discretely* (or that G is a *discrete transformation group*) if the orbit O_y of any point $y \in E$ is a discrete subspace and, moreover, there is a neighbourhood $U(y)$ of the point y called *elementary* hereafter such that the images $g(U)$, $g \in G$ either do not intersect or coincide.

DEFINITION 9. A group G is said to act in E freely (or *without fixed points*) if $g(y) \neq y$ for any $y \in E$, whatever the element $g \in G$, $g \neq e$.

If the discrete transformation group G acts freely then the images $g(U)$ of the elementary neighbourhood $U(y)$ considered above do not coincide for different g.

DEFINITION 10. A transformation group G acting in E discretely and freely is called a *properly discontinuous transformation group.*

Let G be a properly discontinuous transformation group of a space E. Consider the orbit space $E/G = B$ and the natural projection $p: E \to B$ (see Sec. 5, Ch. II).

LEMMA 3. *Let E be a path-connected space, and G a properly discontinuous transformation group in E. Then $p: E \to E/G = B$ is a covering map with the fibre $p^{-1}(b)$, $b \in B$, equal to the orbit O_y of the point y, $p(y) = b$.*

PROOF. By the definition of an orbit space, the projection p is a continuous mapping and $p^{-1}(b) = O_y$ if $p(y) = b$, and, moreover, $O_y \sim G$. The path-connectedness of the space $p(E) = B$ follows from the path-connectedness of E and continuity of p. It remains to construct the coordinate neighbourhoods in B. Let $U(y)$ be an elementary neighbourhood of a point $y \in E$, $b = O_y$ the orbit passing through y and $V(b)$ an open neighbourhood of the point $b \in B$ consisting of all orbits O_z, $z \in U(y)$, passing through the neighbourhood $U(y)$. For a properly discontinuous group G and elementary neighbourhood $U(y)$, we have: $p^{-1}(V) = \bigcup\limits_{g \in G} g(U)$, $g(U)$ being open in E and disjoint. The image of $g(U)$ is a 'sheet' W_g over V of the covering map $p : E \to B$. In fact, $p^{-1}(V) = \bigcup\limits_{g \in G} W_g$, W_g being homeomorphic to V, since the restriction $p_g = p|_{W_g} : W_g \to V$ is a homeomorphism due to the mapping p_g being bijective and open. The neighbourhood $V(b)$ is coordinate, since a homeomorphism $\varphi_V : p^{-1}(V) \to V \times G$ (G being considered with the discrete topology) given on open disjoint sets W_g by the mappings $p_g : W_g \to V \times g$ is defined for any $g \in G$. ∎

EXAMPLES.

8. A covering map $p : S^{2n+1} \to L(k, k_1, \ldots, k_n)$ of the sphere over the generalized lens space determined by the projection of the complex sphere S^n_C (which is homeomorphic to S^{2n+1}) onto the factor space of orbits $L(k, k_1, \ldots, k_n)$ relative to the action of the group Z_k (see Sec. 5, Ch. II). The fibre of this covering coincides with the orbit of the group Z_k, i.e., consists of k elements. Since $\pi_1(S^{2n+1}) = 0$, $\pi_1(L) \sim Z_k$.

9. Consider $E = R^n$ as an Abelian group: it contains a subgroup Z^n of all vectors whose coordinates are integers. The factor group R^n / Z^n equipped with the quotient topology is called an n-dimensional torus T^n. The residue class mapping $p : R^n \to T^n$ is a covering map with the fibre Z^n. Since $\pi_1(R^n) = 0$, we conclude that $\pi_1(T^n) \sim Z^n$.

For the covering map $p : E \to E/G = B$, the following lemma holds.

LEMMA 4. *With the conditions of Lemma 3, the subgroup $N = p_*(\pi_1(E, e_0))$ of the fundamental group $\pi_1(B, b_0)$, where $p(e_0) = b_0$, is a normal subgroup.*

PROOF. Let $[\beta] \in N$, $[\beta_1] \in \pi_1(B, b_0)$. To verify that $[\gamma] = [\beta_1]^{-1} \cdot [\beta] \cdot [\beta_1] \in N$, we lift the path $\gamma = \beta_1^{-1} \cdot \beta \cdot \beta_1$ to the path $\tilde{\gamma} = \tilde{\beta}_1^{-1} \cdot \tilde{\beta} \cdot \tilde{\beta}_1$, where $\tilde{\beta} \in \pi_1(E, e_0)$, $\tilde{\beta}_1$ is from the point e_0 to the point e_1, $p(e_1) = b_0$, $\tilde{\beta}_1^{-1}$ is from the point e_1 to the point e_0. Therefore, $\tilde{\gamma}$ is a loop at the point e_1 and $p\tilde{\gamma} = \gamma$. Since the fibre $p^{-1}(b_0)$ is the orbit O_{e_1} of the group G, there is an element $g_1 \in G$ such that $g_1(e_1) = e_0$. The homeomorphism g_1 maps the loop $\tilde{\gamma}$ to the loop $g_1\tilde{\gamma}$ at the point e_0 so that $[g_1\tilde{\gamma}] \in \pi_1[E, e_0]$. The path $g_1\tilde{\gamma}$ covers the path γ, since the mapping p is constant on the orbits of the group G; therefore, $p(g_1\tilde{\gamma}) = \gamma$ and $[\gamma] = p_*([g_1\tilde{\gamma}])$, i.e., $[\gamma] \in N$. ∎

Coverings whose subgroups $N = p_*(\pi_1(E, e_0))$ are normal are said to be *regular*.

For regular coverings, the family of cosets of the group $\pi_1(B, b_0)$ relative to the subgroup N is a factor group.

Before passing over to calculating $\pi_1(E/G)$, we introduce an important notion of the monodromy group of a covering.

Let $p : E \to B$ be a covering map, and $b_0 \in B$ a certain point of the base space. Let us define the action of the group $\pi_1(B, b_0)$ in the fibre $p^{-1}(b_0) \sim F$. Let $[\beta] \in \pi_1(B, b_0)$, and $e_\alpha \in p^{-1}(b_0)$ be an arbitrary point of the fibre over b_0 whose subscript is an element $\alpha \in F$. Let $\tilde{\beta}$ be a lift of the path β to the point e_α; put $e_{\alpha'} = \tilde{\beta}(1)$, where α' is the subscript of the fibre containing $\tilde{\beta}(1)$. We know already that $\tilde{\beta}(1)$ does not depend on the choice of a path β from the class $[\beta]$, but does only on the class $[\beta]$. Thus, the class $[\beta]$ determines the mapping $\sigma_\beta : p^{-1}(b_0) \to p^{-1}(b_0)$ by the rule $e_\alpha \mapsto e_{\alpha'}$ (or the mapping $\sigma_\beta : F \to F$ by the rule $\alpha \mapsto \alpha'$).

It is easy to see that the mapping σ_β covers the space $p^{-1}(b_0)$.

The following obvious equalities: $\sigma_{\beta_1 \cdot \beta_2} = \sigma_{\beta_2}\sigma_{\beta_1}$, $\sigma_\beta = 1_F$ if $\beta \in e$ (the identity element of $\pi_1(B, b_0)$), $\sigma_{\beta^{-1}} = \sigma_\beta^{-1}$ derived from Lemma 1 on lifting paths signify that the correspondence $\sigma : [\beta] \mapsto \sigma_\beta$ is a representation of the group $\pi_1(B, b_0)$ by 'homeomorphisms', i.e., 'permutations' of the discrete space $p^{-1}(b_0)$ (or F). This representation σ is called a *monodromy of the covering*, and the set of permutations $\{\sigma_\beta\}$, $\beta \in \pi_1(B, b_0)$ the *monodromy group* of the covering.

Thus, the monodromy σ is a homomorphism of the group $\pi_1(B, b_0)$ to the group of all permutations of the fibre.

It follows from Theorem 3 that the point $e_\alpha \in p^{-1}(b_0)$ is fixed for those and only those permutations σ_β for which $[\beta] \in p_*(\pi_1(E, e_\alpha))$. Thus, $p_*(\pi_1(E, e_\alpha))$ is a stability subgroup of the point e_α in the group $\pi_1(B, b_0)$ acting on the fibre $p^{-1}(b_0)$. Moreover, $\sigma_\beta(e_\alpha) = \sigma_{\beta'}(e_\alpha)$ if and only if $[\beta'] \in \{p_*(\pi_1(E, e_\alpha))\}[\beta]$, i.e., to the coset containing the element $[\beta]$ (whence Theorem 4 follows immediately). For different points e_α, $e_{\alpha'}$, the subgroups $p_*(\pi_1(E, e_\alpha))$, $p_*(\pi_1(E, e_{\alpha'}))$ are conjugate with respect to that element $[\beta] \in \pi_1(B, b_0)$ for which $\sigma_\beta(e_\alpha) = e_{\alpha'}$. In fact, if $\tilde{\beta}$ is the corresponding covering path then the correspondence $\gamma \to \gamma' = \tilde{\beta}^{-1} \cdot \gamma \cdot \tilde{\beta}$, where $[\gamma] \in \pi_1(E, e_\alpha)$, establishes an isomorphism between $\pi_1(E, e_\alpha)$ and $\pi_1(E, e_{\alpha'})$ transformed by the monomorphism p_* into the isomorphism

$$p_*(\pi_1(E, e_\alpha)) \to [\beta]^{-1} p_*(\pi_1(E, \alpha))[\beta] = p_*(\pi_1(E, e_{\alpha'})).$$

Let us calculate the monodromy group $\{\sigma_\beta\}$ for the covering map $p : E \to E/G = B$ generated by a properly discontinuous transformation group G.

LEMMA 5. *The monodromy group of the covering map $p : E \to E/G = B$ generated by a properly discontinuous transformation group G of a path-connected space E is isomorphic to G.*

PROOF. Let $e_0 \in E$, $b_0 = p(e_0)$ be base points. We have $p^{-1}(b_0) = O_{e_0}$, where O_{e_0} is the orbit passing through the point e_0 of the group G, i.e., the set of points $\{g(e_0)\}$, $g \in G$. Let $[\beta] \in \pi_1(B, b_0)$ and σ_β the corresponding monodromy transformation. Then $\sigma_\beta(e_0) = g_\beta(e_0)$. Since the path $\tilde{\beta}$ from e_0 to $g_\beta(e_0)$ is carried by the homeomorphism $g \in G$ to the path $g\tilde{\beta}$ from ge_0 to $g(g_\beta, e_0)$, and the path $g\tilde{\beta}$ covers the path β, $\sigma_\beta(ge_0) = g(g_\beta e_0) = g(\sigma_\beta e_0)$.

The correspondence $\sigma_\beta \mapsto g_\beta$ determines a homomorphism of the monodromy group into the group G. In fact, if $\sigma_{\beta_2} \cdot \sigma_{\beta_1}$ is the superposition of σ_{β_1} and σ_{β_2}, then $(\sigma_{\beta_2}\sigma_{\beta_1})e_0 = \sigma_{\beta_2}(g_{\beta_1}e_0) = g_{\beta_1}(\sigma_{\beta_2}e_0) = (g_{\beta_1}g_{\beta_2})e_0$. Therefore $\sigma_{\beta_2} \cdot \sigma_{\beta_1} \mapsto g_{\beta_1} \cdot g_{\beta_2}$.

Furthermore, the permutation $\sigma_\beta^{-1} = \sigma_{\beta^{-1}}$ corresponds to g_β^{-1} and the identity permutation $\sigma_\beta = 1(\beta \in e)$ to $g_\beta = e_G$, the identity element of the group G. We show that the homomorphism $\sigma_\beta \mapsto g_\beta$ is a monomorphism of the monodromy group into the group G. In fact, if $g_\beta = e_G$ then $\sigma_\beta g e_0 = g e_G e_0 = g e_0$ for any $g \in G$, and therefore σ_β is the identity mapping of the fibre O_{e_0}.

The surjectivity of the homomorphism $\sigma_\beta \to g_\beta$ follows from the path-connectedness of E enabling us to join, by a certain path α, the point e_0 to the point $g_* e_0$, where $g_* \in G$ is arbitrary so that α is a lift of the loop $\beta_* = p\alpha$ and $\sigma_{\beta_*} e_0 = \alpha(1) = g_*(e_0)$ therefore $\sigma_{\beta_*} - g_*$. Thus, the isomorphism of the monodromy group with the group G is established. ■

Now it becomes quite easy to prove the basic theorem.

THEOREM 5. *For a covering map $p : E \to E/G = B$ generated by a properly discontinuous transformation group G of a path-connected space E, the factor group of the group $\pi_1(B, b_0)$ relative to the normal subgroup $p_*(\pi_1(E, e_0))$, $p(e_0) = b$, is isomorphic to the group G.*

PROOF. Consider the homomorphism $S : \pi_1(B, b_0) \to G$ given by the composition of the homomorphism of the group $\pi_1(B, b_0)$ into the monodromy group of the covering and isomorphism of the monodromy group to the group G, i.e., the homomorphism given by the correspondence $[\beta] \mapsto \sigma_\beta \mapsto g_\beta$. The inverse image $S^{-1}(e_G)$ consists of those classes $[\beta]$ for which $g_\beta = e_G$, i.e., σ_β is the identity mapping of the fibre $p^{-1}(b_0)$. Therefore, $S^{-1}(e_G) = p_*(\pi_1(E, e_0))$ and the factor homomorphism $\hat{S} : \pi_1(B, b_0)/p_*(\pi_1(E, e_0)) \to G$ is an isomorphism.

COROLLARY. If a covering map $p : E \to E / G = B$ is universal then the group $\pi_1(B)$ is isomorphic to the group G.

We now go back to Examples 6, 7, 8, and 9.

The universal covering $p : R^1 \to S^1$, $p(t) = e^{2\pi i t}$ is generated by a properly discontinuous transformation group by the translation $t \mapsto t + n, n \in Z$ of the axis R^1. Therefore $\pi_1(S^1) \approx Z$ (isomorphism). The monodromy group is also Z and acts on the fibre $F - Z$ by translations $m \to m + n$.

The universal covering $p : S^n \to RP^n, n \geqslant 2$, is generated by a properly discontinuous transformation group Z_2 with the generator $a : S^n \to S^n$ acting by the rule $a(x) = -x$. Therefore, $\pi_1(Rp^n) \approx Z_2$, $n \geqslant 2$. The monodromy group is Z_2 and acts on the fibre $F = p^{-1}(b_0) = \{x_0, -x_0\}$, $x \in S^n$; for the generator σ, we have: $\sigma(x_0) = -x_0$, $\sigma(-x_0) = x_0$, i.e., σ permutes the points of the fibre. The generating element of the group $\pi_1(RP^n, b_0)$ corresponding to the element σ is formed by the homotopy class of the path $p\gamma$, where γ is a path on S^n joining the points x_0 and $-x_0$.

The universal covering $p : S^{2n+1} \to L(k; k_1, \ldots, k_n)$ is generated by a properly discontinuous action of the group Z_k with the generator $a : S^{2n+1} \to S^{2n+1}$. Therefore $\pi_1(L) \approx Z_k$, the monodromy group is also Z_k and acts on the fibre; its generator corresponds to the generator $[\gamma] \in \pi_1(L)$, where γ is the projection of the path in S^{2n+1} joining the point x_0 to the point $a(x_0)$ (find a, $a(x_0)$ using Sec. 5, Item 3, Ch. II).

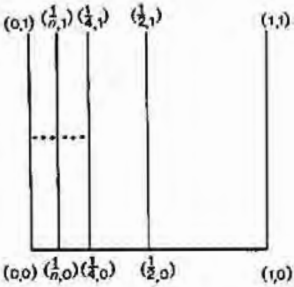


Fig. 91

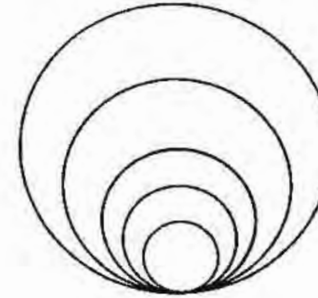


Fig. 92

The universal covering $p : R^n \to T^n$ is generated by a properly discontinuous action of the group Z^n with the generators a_i acting by the rule $(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_n)$, $i = 1, \ldots, n$. Therefore, $\pi_1(T^n) = Z^n$ and the generators $[\gamma_i]$, $i = 1, 2, \ldots, n$ of the group $\pi_1(T^n)$ contain the loops γ_i obtained by the projection p of the paths in R^n joining the point O to the points $a_i(0)$. The monodromy group acts on the fibre $F \sim Z^n$, its generators σ_{a_i}, $i = 1, \ldots, n$ acting on integral vectors from Z^n by the rule: $(k_1, \ldots, k_n) \mapsto (k_1, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_n)$.

To study universal coverings, it is necessary to impose stronger requirements than the path-connectedness on the base space of the covering

DEFINITION 11. A topological space X is said to be *locally path-connected* if for any point $x \in X$, there exists a base of open path-connected neighbourhoods. If neighbourhoods of a base possess, in addition, the property of 1-connectedness, then the space is said to be *locally 1-connected*.

Examples of locally path-connected and locally 1-connected spaces can be given easily (e.g., the Euclidean spaces R^n or manifolds). A locally 1-connected space must not necessarily be 1-connected, an example being the circumference S^1. In Fig. 91, a space is represented (the 'comb space') which is path-connected but does not possess the properties of local path-connectedness (and therefore the local 1-connectedness). Figure 92 illustrating an infinite sequence of circumferences of radii $1/n$, $n = 1, 2, \ldots$, which have a common point of contact, gives an example of a path-connected and locally path-connected but not a locally 1-connected space.

However, for further constructions, it is sufficient to assume the fulfilment of a weaker condition than the local 1-connectedness of a space. This condition is contained in the following definition:

DEFINITION 12. A topological space X is said to be *semi-locally 1-connected* if for any point $x \in X$, there exists a neighbourhood in which any two paths with common ends are homotopic at least on the whole space (or, what is equivalent, in which any loop is contractible at least on the whole space).

It is easy to see that if a space X is locally path-connected and semi-locally 1-connected then at each point $x \in X$, there exists a base of open path-connected neighbourhoods possessing the property for any two paths with common ends in a neighbourhood from this base to be homotopic on the whole space X.

An example of a semi-locally 1-connected, but not locally 1-connected space is the cone over the space drawn in Fig. 92.

Note also that a connected and simultaneously path-connected space is path-connected.

The term 'universal covering' is explained by the fact that a 1-connected space which covers B is a covering space over any other space covering B. More precisely, the following proposition holds.

THEOREM 6. *Let $(\tilde{E}, B, \tilde{F}, \tilde{p})$ be the universal covering over a connected, locally path-connected space B. For any covering (E, B, F, p) over B, there exists a surjective mapping $f : \tilde{E} \to E$ such that the diagram*

$$\begin{tikzcd} \tilde{E} \arrow[rr, "f"] \arrow[dr, "\tilde{p}"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd} \tag{4}$$

is commutative. Moreover, the mapping f is the projection of the covering $(\tilde{E}, E, F', f)$ whose fibre F' is a discrete space which is in bijective correspondence with the group $\pi_1(E)$.

PROOF. We perform it in several stages.

1. A mapping f is constructed in the following way. Let $b_0 \in B$, $e_0 \in p^{-1}(b_0) \subset E$, $\tilde{e}_0 \in \tilde{p}^{-1}(b_0) \subset \tilde{E}$. We construct the mapping f as a lift of the mapping $\tilde{p} : \tilde{E} \to B$ satisfying the condition $f(\tilde{e}_0) = e_0$. For an arbitrary point $x \in \tilde{E}$, consider a path $\gamma : I \to \tilde{E}$ with the origin at $\tilde{e}_0$ and the end at x. According to Lemma 1, there exists a unique lift $\xi_\gamma : I \to E$ of the path $\tilde{p}\gamma : I \to B$. Put $f(x) = \xi_\gamma(1)$. Since the space $\tilde{E}$ is 1-connected, the definition of the mapping f is valid. In fact, any two paths γ, ω in $\tilde{E}$ from x to y are homotopic (with the ends fixed), therefore, their projections $p\gamma$, $p\omega$ are also homotopic in B and the lifts of the latter ξ_γ, ξ_ω (with a common origin) are homotopic in E. According to Lemma 2, ξ_γ and ξ_ω have a common end. The commutativity of diagram (4) is obvious.

The mapping f is continuous and, moreover, a local homeomorphism. This is clear for certain neighbourhoods of the points $\tilde{e}$ and e, viz., sheets $\tilde{W}_\alpha$, W_β lying in $\tilde{E}$, E over a path-connected coordinate neighbourhood V. In fact, for paths γ lying in a neighbourhood of $\tilde{W}_\alpha$, we obtain $\xi_\gamma = (p_\beta^{-1}\tilde{p}_\alpha)\gamma$, therefore the mapping $f \mid \tilde{W}_\alpha = p_\beta^{-1}\tilde{p}_\alpha$ is a local homeomorphism. To verify this fact for any pair of points $x \in \tilde{E}$, $y \in E$, where $f(x) = y$, it suffices to see that x, y may be taken to be new base points $\tilde{e}_0$, e, whereas the mapping f is unaltered (the verification is left to the reader).

2. We now show that f is surjective. Let y be an arbitrary point from E. Consider a path $\gamma : I \to E$ with the origin at e_0 and the end at y. For the path $p\gamma : I \to B$, there exists a unique lift $\eta_\gamma : I \to \tilde{E}$ with the origin at $\tilde{e}_0$ and the end at a certain point $x = \eta_\gamma(1)$. Then the paths $f\eta_\gamma$ and γ possess a common origin and cover the same

path $p\gamma$ in B. Therefore, $f\eta_\gamma(1) = \gamma(1)$, i.e., $f(x) = y$, which means that f is surjective.

3. Show that $f : \tilde{E} \to E$ is the projection of the covering. For an arbitrary point $e \in E$, consider the intersection $\Omega = U \cap V$ of the coordinate neighbourhoods U and V containing the point $p(e)$ for the coverings $(\tilde{E}, B, \tilde{F}, \tilde{p})$ and (E, B, F, p), respectively. The neighbourhood Ω is coordinate for both coverings considered; without loss of generality, it can be assumed to be path-connected. Thus, the commutative diagram arises

$$\begin{tikzcd} \tilde{p}^{-1}(\Omega) \arrow[rr, "f"] \arrow[dr, "\tilde{p}"'] & & p^{-1}(\Omega) \arrow[dl, "p"] \\ & \Omega & \end{tikzcd}$$

Here, the restriction of the mapping f to any sheet $\tilde{W}_\alpha$ from $\tilde{p}^{-1}(\Omega)$ is a homeomorphism

$$f|_{\tilde{W}_\alpha} : \tilde{W}_\alpha \to W_\beta, f_{\tilde{W}_\alpha} = p_\beta^{-1}\tilde{p}_\alpha,$$

where $W_\beta = f(\tilde{W}_\alpha)$ is a sheet from $p^{-1}(\Omega)$.

Consider a sheet W_β containing some point e. Denote the set of those sheets $\tilde{W}_\alpha$, of which the inverse image $f^{-1}(W_\beta)$ consists, by F_β^1; $\tilde{W}_\alpha \in F_\beta^1$ being the connected components of the inverse image $f^{-1}(W_\beta)$. We endow the set F_β^1 with the discrete topology and define the mapping

$$\Psi_{W_\beta} : f^{-1}(W_\beta) \to W_\beta \times F_\beta^1$$

by the formula

$$\Psi_{W_\beta}(x) = (f(x), c(x)),$$

where $c(x)$ is the connected component playing the part of the subscript of the sheet and containing the point $f(x)$. It is obvious that ψ_{W_β} is a local homeomorphism and a bijection and therefore a homeomorphism.

Thereby, for an arbitrary point $e \in E$, a coordinate neighbourhood W_β and the coordinate homeomorphism ψ_{W_β} are constructed (the commutativity of the corresponding diagram is obvious). In view of the note after the definition of a covering, the fibre F_β^1 does not depend, up to bijection, on the choice of a point e and coordinate neighbourhood $W_\beta \subset E$.

4. Thus, $f : \tilde{E} \to E$ is a covering map. Since it is universal, $(\pi_1(\tilde{E}) = 0)$, its fibre F' is in bijective correspondence with the group $\pi_1(E)$. ∎

COROLLARY. Any two universal coverings (E_1, B, F_1, p_1) and (E_2, B, F_2, p_2) over a connected, locally path-connected space B are equivalent, i.e., there exists a homeomorphism $f : E_1 \to E_2$ such that the diagram

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

is commutative.

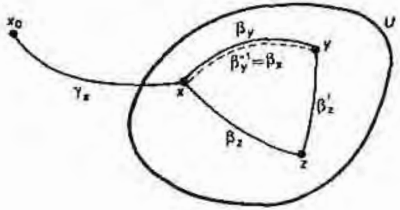


Fig. 93

PROOF. The local homeomorphism established by Theorem 6 is a bijection due to Theorem 4. ∎

We now pass on to the existence theorem for the universal covering.

THEOREM 7. *Let X be a connected, locally path-connected, and semi-locally 1-connected space. Then there exists a universal covering over X.*

PROOF. We recall, first, that if in a base for a covering, a path with fixed ends is homotopized, then the path that covers it is also homotopized with the ends fixed. Therefore, to points e of a 1-connected covering space, there bijectively correspond the homotopy classes of the paths in the base with the origins at the base point x_0 and ends at the projections $p(e)$ of the points e. This property enables us 'to reverse the construction' used in the previous theorem and restore the 1-connected covering space by means of the homotopy classes of the base paths.

Thus, let x_0 be a certain point in X. Consider some homotopy class $[\gamma_x]$ of paths γ_x in X with the origin at the point x_0 and end at some point $x \in X$. The set $\Gamma(x)$ of all such classes, for a certain x, is a fibre over the point x, and the union $E = \bigcup_{x \in X} \Gamma(x)$ of all fibres is the total space. The projection $p : E \to X$ is determined in a natural manner: the projection p associates the class $[\gamma_x]$ with the point x. It is obvious that $p^{-1}(x) = \Gamma(x)$.

We give the first priority to the construction of the topology on E. For each point $[\gamma_x] \in E$, we specify a base of open neighbourhoods $\{\Omega_U([\gamma_x])\}$ as follows. Let U be an arbitrary open, path-connected neighbourhood of the point x. As a neighbourhood of the point $[\gamma_x]$, we take $\Omega([\gamma_x])$, i.e., the set of homotopy classes $[\gamma_y]$ of those paths γ_y from x_0 to $y \in U$, which are the products $\gamma_y = \gamma_x \cdot \beta_y$ of a certain path from the class $[\gamma_x]$ by a path β_y from x to y lying in U; $[\gamma_y]$ depends only on $[\gamma_x]$ and the homotopy class $[\beta_y]$ of the path β_y. The neighbourhood $\Omega_U([\gamma_x])$ is 'open', i.e., a neighbourhood of any of its points: $\Omega_U([\gamma_y]) = \Omega_U([\gamma_x])$ if $[\gamma_y] \in \Omega_U([\gamma_x])$. In fact, $\gamma_y = \gamma_x \cdot \beta_y$, $\gamma_y \cdot \beta_y^{-1} \sim \gamma_x(\beta_y \cdot \beta_y^{-1})$ and since $\beta_y \cdot \beta_y^{-1}$ is a loop at point x homotopic to a constant path, $\gamma_x \sim \gamma_y \cdot \beta_y^{-1}$ (fixed-end homotopy). Since $\beta_y^{-1} = \beta_x$ is a path in U from y to x, we obtain $\gamma_x \sim \gamma_y \cdot \beta_x$. If, now, $[\gamma_z] \in \Omega_U([\gamma_x])$, then $\gamma_z = \gamma_x \cdot \beta_z \sim \gamma_y \cdot (\beta_x \cdot \beta_z)$; if, however, $[\gamma_z] \in \Omega([\gamma_y])$

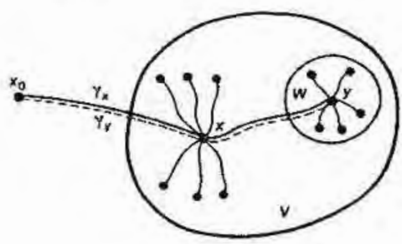


Fig. 94

then $\gamma_z = \gamma_y \cdot \beta_z^1 \sim \gamma_x \cdot (\beta_y \cdot \beta_z^1)$; hence, we conclude that the neighbourhoods $\Omega_U([\gamma_x])$ and $\Omega_U([\gamma_y])$ coincide. The reasoning performed is illustrated by Fig. 93.

Note that due to the semi-local 1-connectedness of the space X, there exists a path-connected open neighbourhood V of the point x for which the homotopy class of the product $\gamma_y = \gamma_x \cdot \beta_y$ does not depend on the choice of a path β from x to y. The neighbourhood V will be the coordinate neighbourhood of the covering under construction. The neighbourhoods Ω_V also form a base for the neighbourhoods of the point γ_x.

Let us now check the continuity of the mapping p. For this, it suffices to verify that $p^{-1}(U)$ is open for any path-connected open neighbourhood U of the point x. Let $[\gamma_y] \in p^{-1}(U)$. Then $[\gamma_y]$ is contained in $p^{-1}(U)$ together with one of its neighbourhoods, viz., the neighbourhood $\Omega_U([\gamma_y])$, i.e., $p^{-1}(U)$ is open.

To show, furthermore, that p is a local homeomorphism, we choose the 'coordinate neighbourhood' V of the point $x \in X$ and the neighbourhood $\Omega_V([\gamma_x])$ of a certain point $[\gamma_x]$ of the fibre $\Gamma(x)$. If $[\gamma_y]$ is an arbitrary point from this neighbourhood then $\gamma_y = \gamma_x \cdot \beta_y$, and all possible paths β_y (from x to y in V), due to the semi-local 1-connectedness of X, fall into the unique homotopy class $[\beta_y]$. Therefore, the correspondence $[\gamma_y] \to y$ specifying the mapping $p : \Omega_V([\gamma_x]) \to V$ is bijective. Moreover, the mapping $p|_{\Omega_V} : \Omega_V([\gamma_x])) \to V$ is a homeomorphism, since it is continuous (as a restriction of the continuous mapping $p : E \to B$ to an open set) and open (since $p(\Omega_W([\gamma_y])) = W$ for any 'coordinate neighbourhood' $W \subset V$ of the point y and $\gamma_y \in \Omega_V([\gamma_x])$ (see Fig. 94).

Thus, p is a local homeomorphism between E and X, and for the 'coordinate neighbourhood' V of the point $x \in X$, we have $p^{-1}(V) = \bigcup_{\alpha \in \Gamma(x)} W_\alpha$, where, for $\alpha = [\gamma_x]$, $W_\alpha = \Omega_V([\gamma_x])$ and $p_\alpha = p : W_\alpha \to V$ are homeomorphisms; each W_α is open in E and path-connected. Moreover, when $\alpha_1 \neq \alpha_2$, $W_{\alpha_1} \cap W_{\alpha_2} = \varnothing$. In fact, if we assume the contrary then there are non-homotopic paths γ_x^1, γ_x^2 and a path γ_z, $z \in V$ such that $[\gamma_z]$ lies in the intersection of the neighbourhoods $\Omega_V([\gamma_x^1])$, $\Omega_V([\gamma_x^2])$. From the previous, $\gamma_z \sim \gamma_x^1 \cdot \beta_z$, $\gamma_z \sim \gamma_x^2 \cdot \beta_z^1$ (fixed-end homotopies). Since $\beta_z^1 \sim \beta_z$ (due to the semi-local 1-connectedness of X), we have $\gamma_z \sim \gamma_x^1 \cdot \beta_z$, $\gamma_z \sim \gamma_x^2 \cdot \beta_z$, i.e., $\gamma_x^1 \cdot \beta_z \sim \gamma_x^2 \cdot \beta_z$; having multiplied both sides of the last rela

tion by β_z^{-1} and taking into account that the loop $\beta_z \cdot \beta_z^{-1}$ is homotopic to a constant, we obtain that $\gamma_x^1 \sim \gamma_x^2$, which is contrary to the original assumption.

Thus, $p^{-1}(V)$ splits into the union of disjoint sheets W_α which are open, path-connected in E (and homeomorphic to V), where α ranges over the fibre $\Gamma(x)$.

It is now natural to define the coordinate homeomorphism $\varphi_V : p^{-1}(V) \to V \times \Gamma(x)$ by assuming the 'coordinates' of the point $[\gamma_y] \in p^{-1}(V)$ to be the 'number' of the sheet W_α to which the point belongs and the point $y \in V$, i.e., the projection of the point γ_y under the homeomorphism $p_\alpha = p|_{W_\alpha} : W_\alpha \to V$. Thus, put $\varphi_V([\gamma_y]) = [y, [\gamma_x]]$ if $[\gamma_y] \in \Omega_V([\gamma_x])$. It is obvious from above that the definition of the mapping φ_V is valid. It remains to show that φ_V is a homeomorphism of an open set $p^{-1}(V)$ in E and the topological product $V \times \Gamma(x)$, where $\Gamma(x)$ is regarded as equipped with the discrete topology.

The bijectivity of φ_V is evident from the above constructions. The continuity of φ_V follows from that of the two mappings $p : p^{-1}(V) \to V$, $p([\gamma_y]) = y$ and $q_V : p^{-1}(V) \to \Gamma(x)$, $q_V([\gamma_y]) = [\gamma_x]$ involved in the definition of φ_V. The continuity of p was ascertained earlier, and that of q_V follows from q_v being locally constant (on each sheet $W_\alpha = \Omega_V([\gamma_x])$). The continuity of φ_V^{-1} is a consequence of the topology of the fibre $\Gamma(x)$ being discrete and $p_\alpha : W_\alpha \to V$ being a homeomorphism. In fact, the base of open neighbourhoods of the point $[y \times \alpha] \subset V \times \Gamma(x)$ is formed by the sets $S(y) \times \alpha$, where $S(y) \subset V$ is a path-connected, open neighbourhood of the point y, and the inverse image $\varphi_V^{-1}(S(y) \times \alpha)$ equals $p_\alpha^{-1}(S(y))$, i.e., an open subset in W_α.

Thus, φ_V is a homeomorphism. The commutativity of the diagram required by the definition of a coordinate homeomorphism is obvious.

Let us verify the path-connectedness of the space E. It suffices to show that an arbitrary point $[\gamma_x]$ from E can be joined to the point $[C_{x_0}]$, i.e., the homotopy class of the constant path (at the point x_0), by a path in E. Let $\gamma_x : I \to E$ be the representative of the class $[\gamma_x]$. We define the path $\zeta^s : I \to X$ for a certain s, $0 \leqslant s \leqslant 1$, by the formula $\zeta^s(t) = \gamma_x(st)$. By associating the number s with the homotopy class $[\zeta^s]$ of the path ζ^s, we obtain the mapping $\omega : I \to E$ satisfying the conditions $\omega(0) = [C_{x_0}]$, $\omega(1) = [\gamma_x]$. The continuity of the mapping ω can easily be established in sufficiently small intervals in $[0, 1]$ whose images fall into the coordinate neighbourhoods $\Omega_V([\gamma_z])$, where $z = \gamma_x(s)$. Therefore, ω is a path in E with the origin at $[\gamma_{x_0}]$ and the end at $[\gamma_x]$ whence it follows that E is path-connected.

Thus, $(E, X, \Gamma(x), p)$ is a covering. To complete the proof of the theorem, we establish the 1-connectedness of the space E. Consider the loop φ of the space E at a point e_0, where e_0 is the homotopy class of the constant mapping C_{x_0}. We show that the loop $\gamma = p\varphi : I \to X$ (at the point x_0) is homotopic to a constant. Note that due to the structure of the space E, for an arbitrary path $\xi : I \to X$ with the origin at x_0 and its unique covering path $\eta : I \to E$ with the origin at e_0, the end $\eta(1)$ of the path η is the homotopy class of the path ξ (in the class of paths with fixed ends). Since φ is a unique path with the origin at e_0 covering the path γ, we obtain that $\varphi(1) = [\gamma] = [C_{x_0}] = e_0$, i.e., the paths γ and C_{x_0} are homotopic with the ends fixed, which implies the contractibility of the loop $\gamma = p\varphi$. Because the projection p induces a monomorphism of fundamental groups, the loop φ is also homotopic to a constant.

Therefore, $\pi_1(E, e_0) = 0$, which completes the proof of the theorem. ■

Note in conclusion, that the condition

$$f_*(\pi_1(X, x_0)) \subseteq p_*(\pi_1(E, e_0))$$

which is necessary for lifting a mapping f is sufficient for a connected, locally path-connected space X. The construction of the lift is in this case based on lifting the paths of the form $f \cdot \alpha$, where α is a path in X with the origin at x_0 and the end at an arbitrary point x. The validity of this construction can be verified by means of the covering homotopy property.

5. **Ramified Coverings.** To conclude this section, we dwell on the notion of ramified covering. An example of a ramified covering (as shown by the example in Sec. 4, Ch. I, of the Riemann surface of the function $w = \sqrt{z}$) may be given by the mapping of the z-sphere S^2 into itself determined by the formula $f(z) = z^2$. It is obvious that the quadruple

$$(S^2 \setminus \{0, \infty\}, S^2 \setminus \{0, \infty\}, Z_2, f)$$

(where Z_2 is a two-point space with the discrete topology) is a covering

DEFINITION 13. A quadruple $(\tilde{M}, M, Z_n, p)$, where $p : \tilde{M} \to M$, is called a *ramified covering* if (i) $\tilde{M}$ and M are two-dimensional manifolds; Z_n a space with the discrete topology and consists of n points; (ii) for a certain finite set $T \subset \tilde{M}$, the quadruple $(\tilde{M} \setminus T, M \setminus p(T), Z_n, p)$ is an n-sheeted covering; (iii) for any point $y \in M$ and its sufficiently small neighbourhood $V(y)$ which is homeomorphic to the disc, the connected components of the set $p^{-1}(V(y))$ are homeomorphic to the disc.

We will call points $x \in T$ the *singular points of the ramified covering.*

Exercise 11°. Show that a Riemann surface P determined by the algebraic function

$$w^n + a_1(z)w^{n-1} + \ldots + a_{n-1}(z)w + a_n(z) = 0,$$

where $a_i(z)$, $i = 1, \ldots, n$, are polynomials (see Sec. 4, Ch. I), is a ramified covering (P, S^2, Z_n, p). Indicate the singular points of this covering. When $n = 2$, compare the result with those obtained in Sec. 4. Ch. I.

Consider an open neighbourhood $V(p(x^i))$ of the image of a singular point x^i such that for all other singular points x^j, it follows from the condition $p(x^j) \in \overline{V(p(x^i))}$ that $p(x^i) = p(x^j)$. The inverse image of the boundary $\partial V(p(x^i))$ of this neighbourhood decomposes into several closed curves, viz., the circumferences which bound the connected components of the set $p^{-1}(V(p(x^i)))$ that are homeomorphic to open discs. Let $U(x^i)$ be the connected component of $p^{-1}(V(p(x^i)))$ containing the point x^i. The degree of the mapping (see Sec. 4, Ch. III)

$$p\,|_{\partial U(x^i)} : \partial U(x^i) \to \partial V(p(x^i))$$

is called the *multiplicity of the branch point* x^i; we will denote it by k_i. It is evident that the multiplicity can be defined also for branch points which are not singular. If $p\,|_{U(x^i)} : U(x^i) \to V(p(x^i))$ is a homeomorphism then, obviously, $\deg p\,|_{\partial U(x^i)} = \pm 1$. In the general case, the generators of $\pi_1(\partial U(x^i))$ and $\pi_1(\partial V(p(x^i)))$ are chosen arbitrarily and so is the sign of k_i. However, in a number of cases, the sign of k_i is determined in a natural way. Thus, for a ramified covering (S^2, S^2, Z_2, z^2), the multiplicity of the points 0 and ∞ is equal to 2, and that of any other point is 1. For the ramified covering $(S^2, S^2, Z_2, \overline{(z^2)})$, the multiplicity of the points 0 and ∞ is equal to -2, and that of any other points is -1.

Exercise 12°. Calculate the multiplicity of the singular branch points of the ramified coverings from Exercise 11.

We now state the following important formula

$$\chi(\tilde{M}) = n \cdot \chi(M) - \sum_i (|k_i| - 1) \tag{1}$$

relating the multiplicities of singular branch points with the Euler characteristics of a space and its base.

We will assume the spaces $\tilde{M}$ and M to be compact and triangulable, i.e., to be closed surfaces. For any singular point $x^j \in T$, we choose a neighbourhood $V(p(x^j))$ as it was done above. Consider now the quadruple

$$(\tilde{M} \setminus \bigcup_j U(x^j), M \setminus \bigcup_j V(p(x^j)), Z_n, p),$$

where x^j ranges over the whole set T. It is obvious that this is an n-sheeted covering (not ramified) whose space and base may be considered triangulable. These triangulations may be chosen sufficiently fine and compatible so that the full inverse images of a vertex, an edge and a triangle from the base are sets of n vertices, edges and triangles, respectively. Therefore, the equality holds:

$$\chi(\tilde{M} \setminus \bigcup_j U(x^j)) = n\chi(M \setminus \bigcup_j V(p(x^j))). \tag{2}$$

Let the full inverse image $p^{-1}(p(x^j))$ consists of m points $x^{j_1}, \dots, x^{j_m}$. Then the full inverse image $p^{-1}(V(p(x^j)))$ consists of m discs $U(x^{j_s})$. Since the boundary $\partial U(x^{j_s})$ is mapped onto $\partial V(p(x^j))$ locally homeomorphically with degree k_{j_s}, $s = 1, \dots, m$, the set $p^{-1}(y) \cap \partial U(x^{j_s})$ consists of precisely $|k_{j_s}|$ points for every point $y \in \partial V(p(x^j))$. Therefore, for every singular point x^j, and points $x^{j_s} \in p^{-1}(p(x_j))$, we have

$$\sum_{s=1}^{m} |k_{j_s}| = n, \tag{3}$$

and the number m of the connected components of the set $p^{-1}(V(p(x^j)))$ satisfies the relation

$$n - \sum_{s=1}^{m} (|k_{j_s}| - 1) = m. \tag{4}$$

We now glue the discs $\tilde{U}(x^{j_s})$ lying over the disc $V(p(x^j))$ to the space $\tilde{M} \setminus \bigcup_j U(x^j)$. Denote the obtained space by $\tilde{M}'$. Since the Euler characteristic of the disc equals 1 and that of its boundary is 0, we obtain

$$\chi(\tilde{M}') = \chi\left(\tilde{M} \setminus \bigcup_j U(x^j)\right) + \sum_{s=1}^{m} \chi(\tilde{U}(x^{j_s}))$$

$$- \sum_{s=1}^{m} \chi\left(\tilde{U}(x^{j_s}) \cap \left(\tilde{M} \setminus \bigcup_j U(x^j)\right)\right) = \chi\left(\tilde{M} \setminus \bigcup_j U(x^j)\right) + m$$

$$= \chi\left(\tilde{M} \setminus \bigcup_j U(x^j)\right) + n - \sum_{s=1}^{m} (|k_{j_s}| - 1). \tag{5}$$

Gluing one by one new discs placed over the remaining points $p(x^l) \in M$, i.e., the projections of the singular points $x^l \in T \subset \tilde{M}$, we obtain:

$$\chi(\tilde{M}) = \chi\left(\tilde{M} \setminus \bigcup_l U(x^l)\right) + ln - \sum_l (|k_l| - 1), \tag{6}$$

where l is the number of different images $p(x^l)$ of the singular points x^l. Remember that

$$\chi\left(\tilde{M} \setminus \bigcup_l U(x^l)\right) \quad \text{and} \quad \chi\left(M \setminus \bigcup_l V(p(x^l))\right)$$

are related by equality (2). Note now that

$$\chi(M) = \chi\left(M \setminus \bigcup_l V(p(x^l))\right) + l, \tag{7}$$

since M may be obtained from $M \setminus \bigcup_l V(p(x^l))$ by gluing l discs to the latter. Thus, from (2), (6) and (7), we find

$$\chi(\tilde{M}) = n\chi(M \setminus \bigcup_l V(p(x^l))) + nl - \sum_i (|k_i| - 1) = n\chi(M) - \sum_i (|k_i| - 1). \ \blacksquare$$

Bringing to mind the expression of the Euler characteristic in terms of the genus of a closed surface, it is easy to derive from formula (1) the sum of the multiplicities of the singular points on $\tilde{M}$ in terms of the genus of M and the genus of $\tilde{M}$. For example, in case M and $\tilde{M}$ are orientable and of genuses p, $\tilde{p}$, respectively, we have

$$\sum_l (|k_l| - 1) = 2(\tilde{p} + n(1 - p) - 1).$$

Exercise 13°. Compare the latter formula with that for the number of branch points of a certain algebraic function, which was established at the end of Sec. 4, Ch. I.

10. SMOOTH FUNCTION ON MANIFOLD AND CELLULAR STRUCTURE OF MANIFOLD (EXAMPLE)

1. An Example of a Function on a Torus. A manifold is a topological space arranged locally as a Euclidean space. However, considered as a whole, it may be quite complicated. The study of the properties of the manifold in the large presents considerable difficulties. How should non-diffeomorphic, non-homeomorphic and homotopy non-equivalent manifolds be distinguished? Are the complex projective space and a sphere of the same dimension homeomorphic? The most coarse of the listed equivalences is homotopy equivalence. Therefore, first of all, the homotopy type of a manifold should be studied. An exceptionally useful instrument in investigating the homotopy type of a manifold and solving many other manifold topology problems is the theory of critical points of smooth functions on manifolds. We illustrate this method by a simple example.

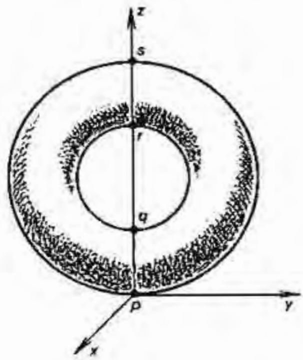


Fig. 95

Consider a two-dimensional torus $M \subset R^3$ touching the plane xy (Fig. 95). Consider a function f on the torus whose value at the point of the torus with coordinates (x, y, z) equals z, the height of the point over the plane xy.

Exercise 1°. Verify that the function f so defined is a smooth function on the torus.

Denote the point of contact of the torus and plane by p, and the points of the torus that lie over p on the perpendicular to the plane by q, r and s in order of increasing height.

While investigating functions on the manifold, we will need the notions of the *Lebesgue set* $(\varphi \leqslant c) = \{x \in X : \varphi(x) \leqslant c\}$ of a function $\varphi : X \to R$ and the *level set* $(\varphi = c) = \{x \in X : \varphi(x) = c\}$ of the function φ. These sets will be used essentially in the analysis of functions on the manifold in Sec. 12.

It is evident that the level line $(f = c)$ of the function $f = z$ is the intersection of the torus with the plane $z = c$. The set of points of the torus lying not higher than the plane $z = c$ is the set $(f \leqslant c)$; we denote it by M^c. The set M^c is empty when $c < 0$; when $c = 0$, M^c consists of one point and is a zero-dimensional manifold; when $0 < c < f(q)$, M^c is homeomorphic to a plane closed disc; when $f(q) < c < f(r)$, this set is homeomorphic to a cylinder; when $f(r) < c < f(s)$, the set M^c is a torus with the cap cut (the cap being homeomorphic to an open plane disc); when $c \geqslant f(s)$, $M^c = M$. All these cases are illustrated in Fig. 96.

Exercises.

2°. Describe the sets $M^{f(q)}$ and $M^{f(r)}$.

3°. Show that when $0 < c < f(s)$ and $c \neq f(q), f(r)$, the set M^c is a two-dimensional manifold with boundary $(f = c)$.

Intuitively, the sets drawn in Fig. 96 are not homeomorphic; nor are they homotopy equivalent. The homotopy type of the set M^c will alter if c passes through the values of the function f at the points p, q, r, s which we singled out. Let us study these changes in more detail.

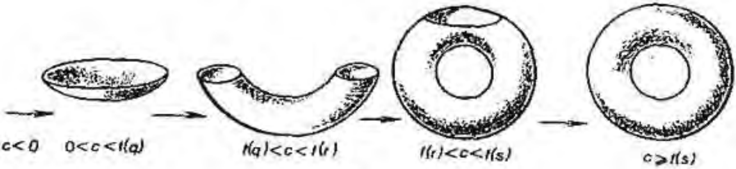


Fig. 96

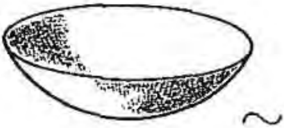


Fig. 97

Fig. 98

Fig. 99

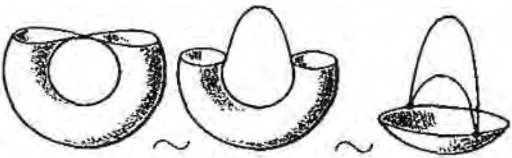


Fig. 100

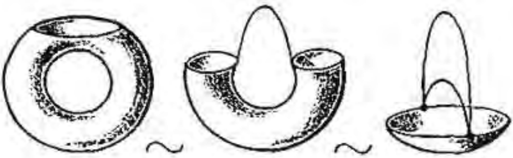


Fig. 101

Obviously, the set M^c when $0 \leqslant c < f(q)$ is homotopy equivalent to the zero-dimensional disc, i.e., a point (Fig. 97); when $c = f(q)$, the set M^c is homotopy equivalent to the disc with an arc handle glued (Fig. 98). Further, when $f(q) \leqslant c < f(r)$, this homotopy type is preserved (Fig. 99). When $c = f(r)$, the set M^c is a cylinder with the boundaries glued to one point and therefore is homotopy equivalent to a cylinder with an arc handle glued or disc with two arc handles (Fig. 100). When $f(r) < c < f(s)$, the set M^c is a torus with the cap removed; it is evident that its homotopy type is the same as in the previous case (Fig. 101). Finally, when $c \geqslant f(s)$, the set M^c is the whole torus obtained from the previous type by gluing the removed cap which is homeomorphic to the plane disc.

Thus, with the use of the smooth function f, we have 'constructed' the torus from discs of different dimensions by consequently gluing the discs and transferring to homotopy equivalent figures.

It will be shown below that any smooth manifold may be obtained in this way.

2. Cell Complexes. Let us analyze the operations of gluing discs in the example considered and describe what means 'to glue a disc' in greater detail.

Let X be a Hausdorff space, and let $\bar{D}^n$ be a closed disc with radius 1 and centre at the origin of the coordinate system in the space R^n, and S^{n-1} its boundary. Let $g : S^{n-1} \to X$ be a continuous mapping. The result of gluing the disc $\bar{D}^n$ to X (relative to the mapping g) is the factor space $X \cup_g \bar{D}^n = (X \cup \bar{D}^n)/R$ of the disjoint union of X and $\bar{D}^n$ with respect to the equivalence R under which $u \sim g(u)$, $u \in S^{n-1}$.

A *cell* is the image e^n of the set int $\bar{D}^n$ in $X \cup_g \bar{D}^n$ relative to the mapping: $X \cup \bar{D}^n \to (X \cup \bar{D}^n)/R$.

Thus, the disc is glued to X along the boundary with the help of the given continuous mapping g.

In the case when $n = 0$, disc $\bar{D}^0$ is a point and its boundary the empty set; the result of gluing $\bar{D}^0$ to X is the disjoint union of X and $\bar{D}^0 = e^0$, i.e., the space X with a point lying separately. Another example: gluing the disc $\bar{D}^n$ to the disc $\bar{D}^0$ produces the sphere S^n.

The discs may be glued one after another, the original space being the cell e^0. Moreover, we shall keep the rule to glue the boundary of the disc D^k to a finite set of cells of dimension not higher than $(k - 1)$. The space which can thus be represented as the result of consequent gluing of discs (of different dimensions) is called a *cell complex*.

For example, the sphere S^n is a cell complex consisting of two cells e^n and e^0, where e^n is glued to e^0 along the boundary.

We now adduce a definition of a cell complex which is independent of the previous considerations and does not involve consequent cell gluing. It is equivalent to the one above.

DEFINITION 1. A *cell complex* is a Hausdorff space K which can be represented as the union $\bigcup_{n=0}^{\infty} \left(\bigcup_{i \in I_n} e_i^n \right)$ of pairwise disjoint sets e_i^n called cells and such that for each cell e_i^n, a mapping $g_i^n : \bar{D}^n \to K$ of a closed ball to the space K is fixed. This

is called a *characteristic mapping*, and the restriction g_i^n to $\operatorname{int} \bar{D}^n = D^n$ is a homeomorphism onto e_i^n. In addition, two axioms must be fulfilled:
(i) *the boundary* $\partial e_i^n = \bar{e}_i^n \setminus e_i^n$ *of each cell* e_i^n *is contained in the union of a finite number of cells of lesser dimensions*;
(ii) *the topology on* K *is such that the set* $A \subset K$ *is closed if and only if for each cell* e_i^n, *the full inverse image* $(g_i^n)^{-1}(A \cap \bar{e}^n) \subset \bar{D}^n$ *is closed in* $\bar{D}^n$.

A cell complex is said to be *finite* if it consists of a finite number of cells.

It should be noted that a space can be decomposed into cells in different ways. The pattern of dividing a cell complex into cells is called a *cellular decomposition*.

Exercises.
4°. Show that the two-dimensional torus is a cell complex.
5°. Show that the sets homotopy equivalent to the Lebesgue sets $(f \leqslant c)$ in the example of Item 1, and drawn in Figs. 97-101 on the right,are cell complexes. Compare their cellular decompositions. Mind the change of the homotopy type of these complexes.

Consider a closed subspace L of a cell complex K. If L is a cell complex whose all cells are also cells of the complex K with the same characteristic mappings, then L is called a *subcomplex of the complex K*.

Exercises.
6°. Let K be a cell complex, L its subcomplex and X a topological space. Let mappings $F : K \to X$ and $f : L \times I \to X$ be such that $f|_{L \times 0} = F|_L$. Show that there exists a mapping $\tilde{F} : K \times I \to X$ such that $\tilde{F}|_{L \times I} = f$ and $\tilde{F}|_{K \times 0} = F$ (the Borsuk homotopy extension theorem).

Hint: Extend the homotopy to each 0-dimensional cell, then to each one-dimensional, etc.

7°. Let K be a cell complex,and L its contractible subcomplex. Show that the spaces K and K/L are homotopy equivalent.
8°. Prove that any cell complex is a normal Hausdorff space.

Mind that in the example considered in Item 1, the homotopy type of the set M^c changed while passing through the values $f(p)$, $f(q)$, $f(r)$ and $f(s)$. The points p, q, r, s differ from the other points of the torus in the following: if in a neighbourhood of any of these points, e.g., of the point p, a local system of coordinates ξ, η on the torus is chosen, then both partial derivatives $\dfrac{\partial f}{\partial \xi}$ and $\dfrac{\partial f}{\partial \eta}$ will vanish at the point p (or at q, r, s, respectively). Such points are called the critical points of the function f; the values of the function at these points are called the critical values of the function f.
Exercise 9°. Using the coordinates of the plane x, y as the local coordinates in neighbourhoods of the points p, q, r, s, show that $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$ at any of these points. Expand the function at these points into a series of powers of x and y up to the terms of the second order inclusive. Mind that the number of minuses attached to the terms of the second degree is precisely the dimension of the cell which should

be glued to M^a to make a transfer to M^b when between the numbers a and b, there is a critical value corresponding to the critical point in question.

11. NONDEGENERATE CRITICAL POINT AND ITS INDEX

1. Nondegenerate Critical Points. Let M^n be a manifold of class C^∞, and $f : M^n \to R^1$ a function of class C^∞.

A point $p \in M^n$ is called a *critical point of the function* f if the equality $\frac{\partial f}{\partial x_1} = \ldots = \frac{\partial f}{\partial x_n} = 0$ holds in the local coordinates $x_1, \ldots, x_n$. The number $f(p)$ is called a *critical value of the function* f. All the remaining points of the manifold M^n are called *noncritical points of the function* f. All numbers which are not critical values of the function f are said to be *noncritical values of this function*.

Exercise 1°. Compare the notions of critical and noncritical values of a function f with those of regular and nonregular values of a smooth mapping (see Sec. 5).

A critical point is said to be *isolated* if there is its neighbourhood such that has no other critical points. A critical point is said to be *nondegenerate* if the matrix of the second partial derivatives $A = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\Big|_p$ is nonsingular. Otherwise, a critical point is said to be *degenerate*.

Consider a quadratic form (Ax, x), where $x \in R^n$. This is called the *Hessian* of the function f at the point p. Since the matrix A is symmetric, the quadratic form (Ax, x) can be reduced to the canonical

$$(Ax, x) = -y_1^2 - y_2^2 - \ldots - y_\lambda^2 + y_{\lambda+1}^2 + \ldots + y_h^2$$

by a convenient choice of the coordinates $y_1, \ldots, y_n$, $h \leqslant n$; if the matrix A is nonsingular, then $h = n$.

The number λ is called the *index of the function f at the point p, and the number $(n - h)$ the degree of singularity of the function f at the point p.*

EXAMPLE. Let us define a function on R^2 by the formula $f(x, y) = x^3 - 3xy^2$. It is obvious that the partial derivatives

$$\frac{\partial f}{\partial x}(x, y) = 3x^2 - 3y^2 \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = -6xy$$

vanish simultaneously only at the point (0, 0) which is thus an isolated critical point. All the second partial derivatives

$$\frac{\partial^2 f}{\partial x^2}(x, y) = 6x, \ \frac{\partial^2 f}{\partial x \partial y}(x, y) = -6y, \ \frac{\partial^2 f}{\partial y^2}(x, y) = -6x$$

equal zero at the point (0, 0). Therefore, the matrix of the second partial derivatives of the function f at the point (0, 0) is zero, and the Hessian of the function f at the point (0, 0) is a quadratic form identically equal to zero. Therefore, the critical point (0, 0) is degenerate, the degree of singularity at the point (0, 0) equals two and the index of f is zero.

Exercises.

2°. Show the correctness (i.e., independence from the choice of a system of local coordinates) of the definitions of a critical point, nondegenerate critical point, the degree of singularity and index of a function at a critical point.

3°. Investigate the critical points of the following functions on R^1 and R^2: (a) $f(x) = x^2$, (b) $f(x, y) = x^3$ and (c) $f(x, y) = x^2y^3$; investigate the critical points of a function on the torus (see Item 1, Sec. 10).

2. The Morse Lemma. A remarkable fact in the theory of critical points is the possibility to represent a function in a neighbourhood of a nondegenerate critical point as a quadratic form and to describe the behaviour of the function by its index.

THEOREM 1 (THE MORSE LEMMA). *Let $f : M^n \to R^1$, and p be a nondegenerate critical point of the function f. Then in a certain neighbourhood U of the point p, there exists a local system of coordinates $y_1, \ldots, y_n$ such that $y_i(p) = 0$, $i = 1, \ldots, n$, and the following identity is valid in U:*

$$f(u) = f(p) - y_1^2 - \ldots - y_\lambda^2 + y_{\lambda+1}^2 + \ldots + y_n^2, \tag{1}$$

where $y_1, \ldots, y_n$ are the coordinates of the point u, and λ is the index of the function f at the point p.

PROOF. If there exists a system of coordinates such that the function f is with respect to it of form (1), then the matrix of partial derivatives $\left(\dfrac{\partial^2 f}{\partial y_i \partial y_j}\right)\Bigg|_p$ is diagonal. The numbers on the diagonal are ± 2, and the number of negative eigenvalues is, on the one hand, equal to the number λ in representation (1), and on the other hand, is the index of f at the point p by definition.

We now prove that such a representation (1) for the function f exists. Let $x_1, \ldots, x_n$ be a local coordinate system such that the point p has the coordinates $(0, \ldots, 0)$. Lemma 1 of Sec. 1 may be applied to the function $f(u) - f(p)$ in a certain neighbourhood U of the point p, whence we obtain the following equality (assuming that $f(p) = 0$)

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x g_i(x_1, \ldots, x_n).$$

Moreover,

$$g_i(0, \ldots, 0) = \frac{\partial f}{\partial x_i}(0, \ldots, 0) = 0,$$

since p is a critical point of f.

Let us apply Lemma 1 of Sec. 1 to the functions g_i again. We obtain

$$g_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j h_{ij}(x_1, \ldots, x_n)$$

and therefore

$$f(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x_1, \ldots, x_n). \tag{2}$$

Denoting $\bar{h}_{ij} = \frac{1}{2}(h_{ij} + h_{ji})$, we obtain $\bar{h}_{ij} = \bar{h}_{ji}$ and

$$f(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} x_i x_j \bar{h}_{ij}(x_1, \ldots, x_n).$$

Since $\bar{h}_{ij}(0, \ldots, 0) = \frac{1}{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(0, \ldots, 0)$, the matrix $(\bar{h}_{ij}(0, \ldots, 0))$ is non-singular. Thus, without loss of generality, the matrix (h_{ij}) in (2) can be regarded as symmetric.

If the functions h_{ij} were constant then, to prove the theorem, it would be sufficient to reduce the quadratic form $f(x_1, \ldots, x_n)$ to the canonical. But, generally, the reasoning should be slightly modified.

For the further theory, it will be convenient to assume additionally that $\frac{\partial^2 f}{\partial x_1^2}(0, \ldots, 0) \neq 0$. This assumption does not lead to loss of generality either, since we can always achieve this by changing the local coordinates (changing the chart). In fact, the quadratic form $\sum_{i,j=1}^{n} h_{ij}(0, \ldots, 0)x_i x_j$ can be reduced, by a linear non-singular change of the coordinates, to such that the element a_{11} of its matrix is not zero. Having performed this coordinate change in formula (2), we will obtain for f (in the new coordinates $x_1', \ldots, x_n'$) a similar representation again

$$f(x_1', \ldots, x_n') = \sum_{i,j=1}^{n} x_i' x_j' h_{ij}'(x_1', \ldots, x_n'),$$

but now, however, $h_{11}'(0, \ldots, 0) \neq 0$.

Thus, assuming that $h_{11}(0, \ldots, 0) \neq 0$, we can write (in a certain neighbourhood of the point $(0, \ldots, 0)$):

$$f(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} h_{ij} x_i x_j = h_{11} x_1^2 + 2 \sum_{i>1}^{n} h_{i1} x_i x_1 + \sum_{i,j>1}^{n} h_{ij} x_i x_j$$

$$= \operatorname{sign} h_{11}(0, \ldots, 0) \left(\sqrt{|h_{11}|}\, x_1 + \sum_{i>1}^{n} \frac{h_{i1}}{\operatorname{sign} h_{11}(0, \ldots, 0)\sqrt{|h_{11}|}} x_i \right)^2$$

$$-\frac{1}{|h_{11}|}\sum_{i,j>1}^{n} h_{i1}h_{1j}x_ix_j + \sum_{i,j>1}^{n} h_{ij}x_ix_j$$

$$= \operatorname{sign} h_{11}(0, \ldots, 0)y_1^2 + \sum_{i,j>1}^{n}\left(h_{ij} - \frac{h_{i1}h_{j1}}{|h_{11}|}\right)x_ix_j,$$

where the new coordinate y_1 depends on $x_1, \ldots, x_n$ smoothly:

$$y_1 = \sqrt{|h_{11}(x_1, \ldots, x_n)|}\,x_1 + \sum_{i>1}^{n}\frac{h_{i1}(x_1, \ldots, x_n)x_i}{\operatorname{sign} h_{11}(0, \ldots, 0)\sqrt{|h_{11}(x_1, \ldots, x_n)|}}.$$

Applying the inverse mapping theorem (see Sec. 1), we shall see that the transformation $(x_1, x_2, \ldots, x_n) \to (y_1, x_2, \ldots, x_n)$ is a diffeomorphism in a neighbourhood of the point $(0, \ldots, 0)$.

Note, further, that the matrix

$$\left(h_{ij} - \frac{h_{i1}h_{j1}}{|h_{11}|}\right), \quad 1 < i, j \leqslant n$$

is nonsingular at the point $(0, \ldots, 0)$ and symmetric (verify!). Therefore, we can apply the above reasoning to the function

$$\sum_{i,j>1}^{n}\left(h_{ij} - \frac{h_{i1}h_{1j}}{|h_{11}|}\right)x_ix_j$$

and so on, as in the classical Lagrange algorithm for reducing a quadratic form to the canonical. Finally, we come to an expression of form (1) for the function f. ■

Exercises.

4°. Prove that any nondegenerate critical point is isolated.

5°. Find representations (1) defined by the Morse lemma for the height function on the torus (see Sec. 10) at critical points.

6°. Prove that the points of maximum and minimum of a smooth function on a manifold without boundary are critical. Calculate the indices at the points of maximum and minimum if the points are known to be nondegenerate.

3. The Gradient Field. Let $A_x(u, v)$ be the Riemannian metric on M^n. For any point $x \in M^n$, we choose a vector $y_x \in T_xM^n$ so that the following condition may be fulfilled: for an arbitrary vector $l_x \in T_xM^n$, the following equality is valid

$$A_x(y_x, l_x) = (df)_x(l_x), \tag{3}$$

where $(df)_x(l_x)$ is the value of the differential of the function f at the point x on the vector l_x.

The field y_x obtained is called the *gradient field* of the function f and denoted by $\operatorname{grad} f(x)$.

Exercises.

7°. Show that in the local coordinates, the gradient field is of the form

$$\operatorname{grad} f(x) = \left(x_1, \ldots, x_n, \left(\sum_{i=1}^{n} a^{i1}(x) \frac{\partial f}{\partial x_i}\right) \frac{\partial}{\partial x_1}, \ldots, \left(\sum_{i=1}^{n} a^{in}(x) \frac{\partial f}{\partial x_i}\right) \frac{\partial}{\partial x_n}\right),$$

where $a^{ij}(x)$ are the coefficients of a matrix which is inverse to the matrix $(a_{ij}(x))$ of the form $A_x(u, v)$.

8°. Prove that for a function of class C^∞, the gradient field is a smooth vector field.

9°. Prove that $\operatorname{grad} f(x^0) = 0$ if and only if x^0 is a critical point of the function f.

12. DESCRIBING HOMOTOPY TYPE OF MANIFOLD BY MEANS OF CRITICAL VALUES

In this section, the homotopy type of a manifold will be described by means of critical values of a smooth function on the manifold. Such a description was first given by Morse. We will show that a compact manifold is homotopy equivalent to a cell complex. Some details of the proof (and in a number of cases, rather subtle) will be omitted.

1. The Structure of the Lebesgue Sets of Smooth Functions. Let M be a compact n-dimensional C^∞-manifold, f a function of class C^∞ on M, whose all critical points are nondegenerate. For any number a, the set $(f < a)$ is an open subset in M, and therefore a submanifold in M. Assume now that a is a noncritical value of f and $f^{-1}(a) \neq \varnothing$. Let us show that the set $M^a = (f \leqslant a)$ is a manifold with the boundary $(f = a)$. Let $u \in f^{-1}(a)$. By the theorem on rectifying a mapping (see Sec. 1), the function f can be represented in a certain neighbourhood of the point u in local coordinates as the projection π of the space R^n onto the straight line R^1 (Fig. 102). The inverse image of the point a under this projection is the subspace R^{n-1}, i.e., the boundary of the half-space R^n_-. At points of the half-space, the function f assumes values which are not greater than a. This means that there exists a neighbourhood of the point u in M^a which is homeomorphic to the half-space.

Therefore, $M^a = (f \leqslant a)$ is an n-dimensional manifold whose boundary is the $(n - 1)$-dimensional manifold $(f = a)$.

2. The Conditions for the Homotopy Equivalence of Lebesgue Sets. Let a and b be noncritical values of a function f, and the line-segment $[a, b]$ contain no critical values. We will shift the sets $(f = c)$ through the set $(f = a)$ along the lines orthogonal to the level manifolds $(f = c)$, $a \leqslant c \leqslant b$ (Fig. 103). Thus, we specify a deformation $\varphi^b_a(t)$, $a \leqslant t \leqslant b$, of the manifold M^b onto the

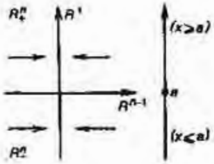


Fig. 102

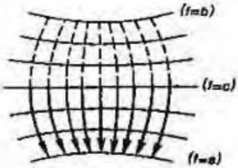


Fig. 103

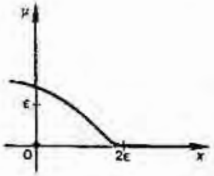


Fig. 104

manifold M^a. Therefore, M^a is a strong deformation retract of M^b, and hence M^a and M^b are homotopy equivalent.

A rigorous proof of the existence of the mapping φ_a^b should include the construction of lines orthogonal to the level manifolds. They can be defined as integral curves of the vector field $X(u)$, where the vector $X(u) \in T_uM$ is determined from the condition $<X(u), h> = 0$ for all $h \in T_u$ $(f = c = f(u))$, i.e., from the condition for the orthogonality of the vector $X(u)$ to the space tangent to the level manifold $(f = c)$. The symbol $<,>$ denotes the Riemannian metric which can be always introduced on the manifold (see Sec. 6). For a level submanifold to be reduced to a level submanifold at any moment t, we define the vector field $X(u)$ by the formula

$$X(u) = \rho(u) \quad \text{grad} \quad f(u),$$

where $\rho(u)$ is a smooth function on M whose values are equal to $1/<\text{grad} f(u), \text{grad} f(u)>$ on $M^b \setminus M^a$ and zero outside a certain neighbourhood of $M^b \setminus M^a$ not containing critical points.

The definition of a deformation of M^b onto M^a may also be valid in the case of a being a critical value of f.

3. The Change of the Homotopy Type while Passing through a Critical Value. Thus, the homotopy type of the set M^c is unaltered if the number c, while increasing (or decreasing), does not assume a critical value c_0. Let us see now what is taking place when c does assume this value.

The proof of the following useful statement is left to the reader.

Exercise 1°. Prove that a smooth function in a compact manifold whose all critical points are nondegenerate possesses a finite number of critical points and critical values.

Consider the critical value c_0. Assume that there corresponds to it a unique critical point p, $f(p) = c_0$. We choose a neighbourhood U of the point p and the local coordinates specified by the Morse lemma (see Sec. 11), where the function f is represented in these coordinates $y_1, \ldots, y_n$ in the form

$$f(u) = c_0 - y_1^2 - \ldots - y_\lambda^2 + y_{\lambda+1}^2 + \ldots + y_n^2.$$

We choose ε such that the set $[c_0 - \varepsilon, c_0 + \varepsilon]$ does not contain other critical points

and the point with the local coordinates $(y_1, \ldots, y_n)$, $\sum_{i=1}^{n} y_i^2 \leqslant 2\varepsilon$, belongs to U.

We construct a smooth function F in M so that it may differ from f only in U, the sets $(f \leqslant c_0 + \varepsilon)$ and $(F \leqslant c_0 - \varepsilon)$ being homotopy equivalent. This being done, we compare the sets $(f \leqslant c_0 - \varepsilon)$ and $(F \leqslant c_0 - \varepsilon)$. This happens to be more convenient than comparing the sets $(f \leqslant c_0 - \varepsilon)$ and $(f \leqslant c_0 + \varepsilon)$ directly. To construct the function F, a smooth function μ on R^1 is required, such that possesses the following properties:

$$\mu(0) > \varepsilon,\ \mu(x) = 0 \text{ when } x > 2\varepsilon,$$
$$-1 < \mu'(x) \leqslant 0 \text{ when } -\infty < x < \infty.$$

The form of the graph of the function μ satisfying these properties is shown in Fig. 104.

Exercise 2°. Give an example of a function μ possessing the indicated properties.

Let us specify the smooth function F by the formulae

$$F(v) = \begin{cases} f(v) & \text{when } v \,\overline{\in}\, U, \\ f(v) - \mu\left(\sum_{i=1}^{\lambda} y_i^2 + 2 \sum_{i=\lambda+1}^{n} y_i^2\right) & \text{when } v \in U. \end{cases}$$

It is easy to see that the critical points of the function F coincide with the critical points of the function f (although $f(p) \neq F(p)$).

To a critical point p of the function F, there corresponds the critical value $F(p) = c_0 - \mu(0) < c_0 - \varepsilon$. Since the value of the function F coincides with those of the function f at other critical points, the line-segment $[c_0 - \varepsilon, c_0 + \varepsilon]$ contains no critical values of F. Consequently, the set $(F \leqslant c_0 - \varepsilon)$ is a strong deformation retract of the set $(F \leqslant c_0 + \varepsilon)$. But $(F \leqslant c_0 + \varepsilon) = (f \leqslant c_0 + \varepsilon)$. Therefore, $(F \leqslant c_0 - \varepsilon)$ is a strong deformation retract of the set $(f \leqslant c_0 + \varepsilon)$. Thus, these sets are homotopy equivalent.

We will further compare the homotopy types of the sets $(f \leqslant c_0 - \varepsilon)$ and $(F \leqslant c_0 - \varepsilon)$ (instead of comparing the homotopy types of the sets $(f \leqslant c_0 - \varepsilon)$ and $(f \leqslant c_0 + \varepsilon)$). Denote the closure of the set $(F \leqslant c_0 - \varepsilon) \setminus (f \leqslant c_0 - \varepsilon)$ by H. Consider the cell e^λ consisting of those points $u \in U$ whose coordinates $y_1, \ldots, y_n$ satisfy the conditions $\sum_{i=1}^{\lambda} y_i^2 < \varepsilon$, $\sum_{i=\lambda+1}^{n} y_i^2 = 0$. The cell e^λ lies inside H; it is glued to the set $(f \leqslant c_0 - \varepsilon)$ along the set of those points u for which $\sum_{i=1}^{\lambda} y_i^2 = \varepsilon$.

A neighbourhood of a critical point of index 1 on a two-dimensional manifold (e.g. the point q from the example of Sec. 10) is drawn in Fig. 105; the set $M^{c_0 - \varepsilon} = (f \leqslant c_0 - \varepsilon)$ is shaded, the set H is shaded twice, the cell e^λ is denoted by a thick line.

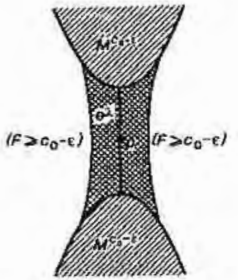


Fig. 105

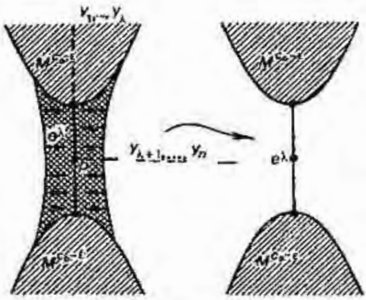


Fig. 106

Let us specify the deformation Γ_t of the set $(F \leqslant c_0 - \varepsilon) = M^{c_0 - \varepsilon} \cup H$ onto the set $M^{c_0 - \varepsilon} \cup e^\lambda$ by letting Γ_t be the identity mapping onto $M^{c_0 - \varepsilon}$, and be defined on H by the formula

$$\Gamma_t(y_1, \ldots, y_n) = \begin{cases} y_1, \ldots, y_\lambda, ty_{\lambda+1}, \ldots, ty_n \text{ for } & \sum\limits_{i=1}^{\lambda} y_i^2 \leqslant \varepsilon, \\ y_1, \ldots, y_\lambda, s_t y_{\lambda+1}, \ldots, s_t y_n \text{ for } & \varepsilon < \sum\limits_{i=1}^{\lambda} y_i^2 < \sum\limits_{i=\lambda+1}^{n} y_i^2 + \varepsilon, \end{cases}$$

where $s_t = t + (1 - t)\sqrt{\dfrac{\sum\limits_{i=1}^{\lambda} y_i^2 - \varepsilon}{\sum\limits_{i=\lambda+1}^{n} y_i^2}}$, $0 \leqslant t \leqslant 1$.

This deformation is shown by arrows in Fig. 106.

Exercise 3°. Verify the correctness of the definition of the deformation Γ_t.

Thus, the set $(f \leqslant c_0 - \varepsilon) \cup e^\lambda$ is a strong deformation retract of the set $(F \leqslant c_0 - \varepsilon)$, and therefore of the set $(f \leqslant c_0 + \varepsilon) = M^{c_0 + \varepsilon}$. Thus, $M^{c_0 + \varepsilon}$ is of the same homotopy type as the set $M^{c_0 - \varepsilon} \cup e^\lambda$, i.e., of the set $M^{c_0 - \varepsilon}$, with the cell glued in a special manner* and of dimension equal to the index of the critical point corresponding to the value c_0.

We considered the case when to a critical value of a function there corresponded a unique critical point. Consider now the general case.

* We omit the gluing mappings in notation.

Exercise 4°. Construct a smooth function in a two-dimensional manifold such that all its critical points are nondegenerate and to one critical value there correspond several critical points.

Let to a critical value c_0, there correspond $k > 1$ critical points. All the constructions described above can be performed simultaneously in a neighbourhood of each critical point. The set $M^{c_0+\varepsilon}$ has the homotopy type of the set $M^{c_0-\varepsilon} \cup e^{\lambda_1} \cup \ldots \cup e^{\lambda_k}$, i.e., the set $M^{c_0-\varepsilon}$ with the cells e^{λ_i} glued to it in a special manner, the dimension λ_i being equal to the index of the i-th critical point corresponding to c_0.

Let c' be the least of the critical values which are greater than c_0, and let there be no other critical values in the ε-neighbourhoods of c_0 and c'. Let to the value c', there correspond k' critical points with indices $\lambda'_1, \ldots, \lambda'_{k'}$. The set $M^{c_0-\varepsilon} \cup e^{\lambda_1} \cup \ldots \cup e^{\lambda_k}$ is homotopy equivalent to the set M^a for $c_0 \leqslant a < c'$. The set $M^{c'}$ is, in turn, homotopy equivalent to the set $M^a \cup e^{\lambda'_1} \cup \ldots \cup e^{\lambda'_{k'}}$.

Let us establish the homotopy equivalence of the sets

$$M^{c'} \text{ and } (M^{c_0-\varepsilon} \cup e^{\lambda_1} \cup \ldots \cup e^{\lambda_k}) \cup e^{\lambda'_1} \cup \ldots \cup e^{\lambda'_{k'}}.$$

To this end, we deform the set $M^a \cup e^{\lambda'_1} \cup \ldots \cup e^{\lambda'_{k'}}$ onto the set

$$(M^{c_0-\varepsilon} \cup e^{\lambda_1} \cup \ldots \cup e^{\lambda_k}) \cup e^{\lambda'_1} \cup \ldots \cup e^{\lambda'_{k'}},$$

using the constructed deformation of M^a onto $M^{c_0-\varepsilon} \cup e^{\lambda_1} \cup \ldots \cup e^{\lambda_k}$.

Exercise 5°. Investigate how the cells $e^{\lambda'_1}, \ldots, e^{\lambda'_{k'}}$ are glued to the set $M^{c_0-\varepsilon} \cup e^{\lambda_1} \cup \ldots \cup e^{\lambda_k}$.

We underline that gluing a cell is performed, at each stage, not in an arbitrary, but in a strictly definite manner (up to the homotopy class of the mapping of the sphere, i.e., the boundary of the cell to the corresponding space). Therefore, gluing a cell is determined by an element of the homotopy group of the corresponding space; the dimension of this group equals the dimension of the cell less one.

4. The Homotopy Type of a Manifold. Here, we outline the construction of a cell complex homotopy equivalent to a manifold M just like it was done in Sec. 10 for the torus.

Let c_1 be the least critical value of a function f. It is obvious that for $a < c_1$, the set $(f \leqslant a)$ is empty. Since c_1 is the least critical value, all critical points corresponding to c_1 are the points of minimum; their indices equal zero. The set $(f \leqslant c_1)$ consists of a finite number of points; it can be regarded as the one obtained by gluing several cells of dimension zero to the empty set.

Let c_2 be another critical value which is next in magnitude. When $c_1 < c < c_2$, the set $(f \leqslant c)$ is obtained by 'inflating' the points from $(f \leqslant c_1)$; it consists of a finite number of sets homeomorphic to the n-dimensional disc, and homotopy equivalent to the set $(f \leqslant c_1)$. The set $(f \leqslant c_2)$ is homotopy equivalent to the set $(f \leqslant c_1)$ with cells of different (generally speaking, of any from 0 to n) dimensions equal to the indices of critical points corresponding to c_2, glued to it. Obviously, the latter set is a cell complex.

Having taken a critical value c_3, which is next in magnitude, we obtain that $(f \leqslant c_3)$ is homotopy equivalent to the result of consequently gluing to $(f \leqslant c_1)$ the cells corresponding to the critical points with the critical value c_2, and then the cells

corresponding to the critical points with the critical value c_3. Such a space can be made a cell complex by adjusting the boundary mappings of the cells being glued.

Exercise 6°. Prove that each mapping of the sphere S^m to a cell complex K is homotopic to a mapping of the sphere to a subspace K^m of the space K,consisting of cells of dimension less than or equal to m.

In the general case, the set $M^a = (f \leqslant a)$ when $a \geqslant \max_{u \in M} f(u)$ is homotopy equivalent to the space which is a cell complex obtained from the empty set by consequently gluing cells corresponding to the critical points with the critical values c_i, in order of increasing c_i, $-\infty < c_i < a$.

Note that if c_r is the greatest critical value, then the critical points at which the value of the function f equals c_r are the points of maximum and hence their indices equal the dimension of the manifold M.

We now formulate the final statement.

THEOREM 1. *Each smooth function f in a compact manifold M having only nondegenerate critical points defines a homotopy equivalence of the manifold M with a certain finite cell complex whose cells are in one-to-one correspondence with critical points of the function f, the dimension of the cell being equal to the index of the corresponding critical point.*

We now dwell on the existence of a smooth function in a compact manifold having only nondegenerate critical points. Such a function may be constructed in the following way. Consider an embedding of the manifold M into the Euclidean space R^l of a sufficiently large dimension l. We define the function f by the formula $f(p) = (u - p, u - p)$, where (,) is the scalar product, u a fixed vector in R^l and $p \in M \subset R^l$. Using the Sard theorem (see Sec. 5), we can show that there exists a vector $u \in R^l$ such that the function f has only nondegenerate critical points.

This result enables us to make the following important conclusion:

THEOREM 2. *Any compact,smooth manifold has the homotopy type as a finite cell complex.*

FURTHER READING

To the reader who starts studying smooth manifold theory, first of all, *Topology from the Differential Viewpoint* [55] by Milnor, *Differential Topology. First Steps* [81] by Wallace, *Modern Geometry* [28] by Dubrovin et al., *A Course of Differential Geometry and Topology* [58] by Mishchenko and Fomenko, *Differential Topology* [41] by Hirsch, *Elementary Differential Topology* [60] by Muncres and *Introduction to Morse Theory* [67] by Postnikov, as those most suitably expounding the theory, can be recommended. To study manifold theory and its applications further, the reader can also be recommended the classical monograph *Smooth Manifolds and Their Applications to Homotopy Theory* [66] by Pontryagin, *Variétés différentiables. Formes courantes, formes harmoniques* [69] by De Rham, *Morse Theory* [54] by Milnor, the fundamental *First Course of Topology. Geometric Chapters* [70] by Rohlin and Fuchs and *Stable Mappings and*

Singularities [37] by Golubitzky and Guillemin. The basic concepts of smooth topology are introduced efficiently in *Outline of Topology of Manifolds* [21] by Chernavsky and Matveyev. While studying Ch. IV, *Problems in Geometry* [61] by Novikov et al., and *Problems in Differential Geometry and Topology* [59] by Mishchenko et al. may prove useful.

As regards Secs. 1 and 2, the textbooks by Shilov *Mathematical Analysis. Functions of Several Real Variables* [72] (Parts I and II), Spivak *Calculus on Manifolds* [74] (as far as Sec. 1 is concerned) and Bröcker and Lander *Differential Germs and Catastrophes* [19] (Sec. I) may prove useful.

It will be useful to see the proofs of the theorems on embedding a manifold into a Euclidean space and on the set of nonregular values of a smooth mapping in the books by Pontryagin [66] (Ch. I, Secs. 2 and 3) and Postnikov [67] (Ch. V, Sec. 6).

Applications of the notion of tangent bundle to mechanics may be seen in *The Mathematical Foundations of Quantum Mechanics* [50] (Ch. I) by Mackey and *Mathematical Methods in Classical Mechanics* [9] (Ch. IV) by Arnold.

As far as vector fields on a manifold are concerned, the reader is referred to the book by Postnikov [67] (Ch. IV, Sec. 6) and *Elementary Topics in Differential Geometry* [80] by Thorpe. It is useful to get acquainted with applications of this theory to differential equations by *Ordinary Differential Equations* [10] (Ch. 5) by Arnold.

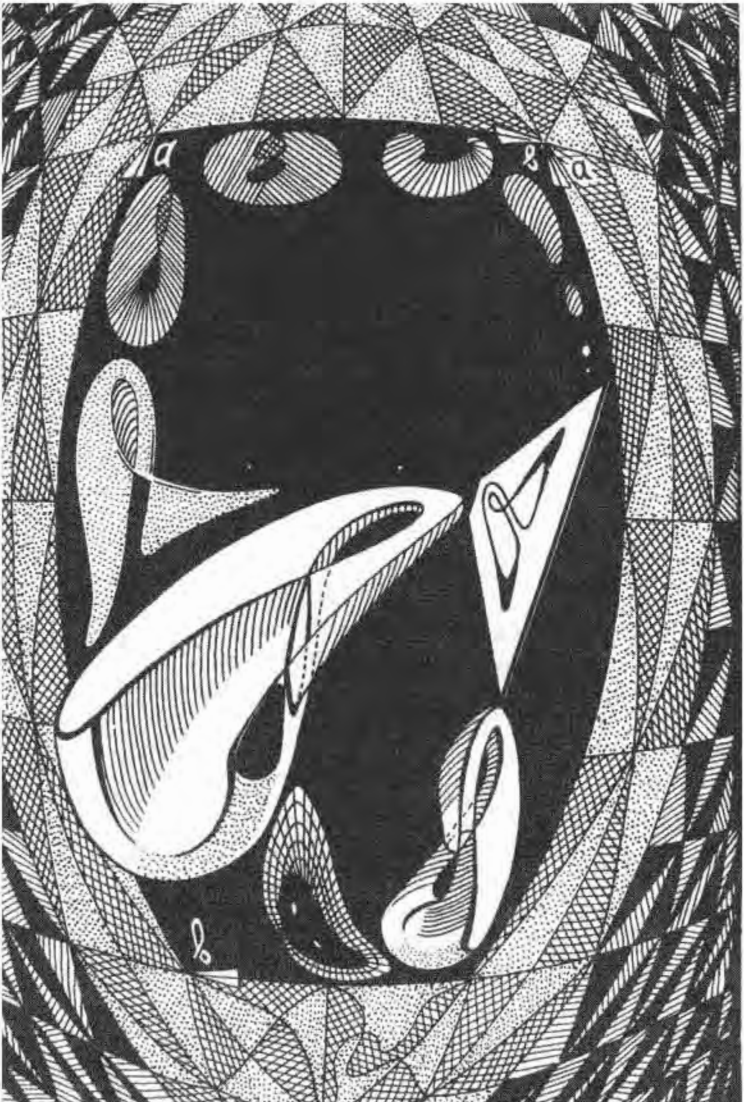
The theory of coverings is expounded quite comprehensibly for the beginner in *Algebraic Topology: An Introduction* [52] by Massey. Quite comprehensive introductions to fibre bundle theory and the theory of coverings may be found in *Homotopy Theory* [43] (Ch. III) by Hu S.-T., *Homotopy Theory* [33] (Ch. I, Secs. 5 and 7) by Fuchs et al., and *First Course of Topology* [70] (Ch. 70) by Rohlin and Fuchs. Ramified coverings are described in detail in *Introduction to Riemann Spaces* [75] by Springer and *Riemannsche Flächen* [31] by Forster. Note that a great number of interesting and useful problems on the theory of coverings and ramified coverings are contained in *Modern Topics of Topology of Manifolds* (Preparatory course: elements of topology) [20] by Chernavsky and Matveyev.

To study the theory of critical points of functions in manifolds (Secs. 10-12), the reader is recommended the above-mentioned books by Milnor [55], Wallace [81], Postnikov [67] (Ch. VI, Secs. 1-4), Golubitzky and Guillemin [37], Hirsch [41], and also the monograph *Morse Theory* [54] (Ch. I) by Milnor. The development of the theory of critical points of smooth functions is the theory of singularities of smooth mappings. The books by Bröcker and Lander [19], Golubitzky and Guillemin [37], *Catastrophe Theory* by Arnold [8], *Catastrophe Theory and Its Applications* [68] by Poston and Stewart may serve as an introduction to this rapidly developing branch of modern mathematics. The present-day state of singularity theory may be acquainted with by the monograph *Singularities of Differential Mappings* [11] by Arnold et al.

V

Homology Theory

In this chapter, the homology groups will be defined for any topological space. The idea of constructing homology groups is, as was already mentioned, due to H. Poincaré. The useful idea of the reduction of topological problems to algebraic was realized, for the first time in the history of topology, by the construction of homology groups and the fundamental group. Homology theory still remains topologically basic. Almost all topological invariants, from homotopy groups to special invariants of fibre bundles, are expressed, finally, in terms of invariants of homology groups. This circumstance is due to a better calculability of homology groups, though to define homology groups is somewhat more complicated than for example, homotopy groups.


a
b
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b

1. PRELIMINARY NOTES

Let us illustrate the idea of reasoning that has lead to the notion of homology. In studying two-dimensional manifolds, we often distinguish intuitively between non-homeomorphic manifolds. However, in studying manifolds of higher dimensions, geometric intuition proves less effective. To distinguish between non-homeomorphic manifolds of high dimensions, we may attempt to apply the following idea. Let M_1^n, M_2^n be two n-dimensional manifolds. We will consider compact submanifolds * in M_1^n, M_2^n. If any q-dimensional submanifold ($q < n$) in M_1^n is the boundary of a $(q + 1)$-dimensional submanifold in M_1^n, and there is a q-dimensional submanifold in M_2^n which is not the boundary of a submanifold in M_2^n, then the manifolds in M_1^n and M_2^n are necessarily non-homeomorphic. Thus, any 1-dimensional submanifold (compact) of the sphere S^2 is a boundary, whereas on the torus $T^2 = S^1 \times S^1$, it is easy to indicate circumferences which are not boundaries of any two-dimensional submanifold in T^2 (Fig. 107).

If, however, there are submanifolds both in M_1^n and M_2^n which are not boundaries, then we may try to compare the 'quantity' of such manifolds in M_1^n and M_2^n. Consider the set $\{V_\alpha^q\}$ of all q-dimensional cycles, i.e., q-dimensional submanifolds (without boundary) of the manifold M^n. Let W^{q+1} be a submanifold of M^n with a boundary consisting of connected manifolds $V_1^q, \ldots, V_m^q$, $V_i^q \in \{V_\alpha^q\}$. We will say in this case that the cycle $V_1^q + \ldots + V_m^q$ is homologous to zero. Thus, an equivalence relation is introduced on the set $\{V_\alpha^q\}$: two cycles are equivalent (homologous) if they are different by a cycle which is homologous to zero; the equivalence classes of q-dimensional cycles are called q-dimensional *homology groups of the manifold* M^n.

If there is an integer q such that there are more q-dimensional homology groups of the manifold M_1^n than q-dimensional homology groups of the manifold M_2^n, then this means that M_1^n and M_2^n are non-homeomorphic.

The notion of the sum of two disjoint cycles has been introduced for the set $\{V_\alpha^q\}$. This does not mean, however, that the group structure has been introduced on it. Therefore we cannot assume, for the present, that homology groups form a group. The visual definition of homology groups given above is, however, inconvenient for calculations. It is more efficient to consider the cycles (manifolds) made up

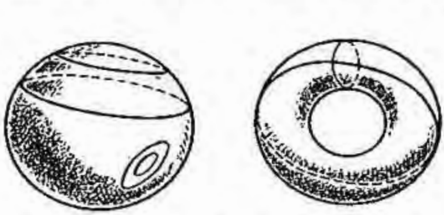


Fig. 107

* Subspaces which are C^0-manifolds.

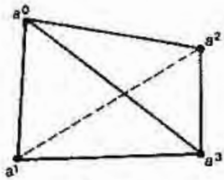


Fig. 108

of certain elementary manifolds with boundaries. How this can be done is shown in the following example.

Let Π^2 be the surface of a tetrahedron (Fig. 108); it is obvious that Π^2 is homeomorphic to the sphere S^2. We will consider the 0-dimensional manifolds which consist of the vertices of the tetrahedron, 1-dimensional manifolds consisting of its edges and 2-dimensional manifolds that consist of its faces and admit a boundary for 1-dimensional and 2-dimensional manifolds; it is natural to treat set-theoretic addition of two manifolds as the sum. To use this for the algebraization of the objects in question, we consider the group of formal linear combinations * of the vertices with integral coefficients (the group of 0-dimensional chains), edges (the group of 1-dimensional chains) and faces (the group of 2-dimensional chains). Moreover, for every edge, we fix an order of vertices (a^i, a^j) and identify $(-1)(a^i, a^j)$ with (a^j, a^i); we fix the direction of circumnavigating the vertices (a^i, a^j, a^k) for each face and identify $(-1)(a^i, a^j, a^k)$ with (a^j, a^i, a^k).

Let us now define the boundary of the edge (a^i, a^j) as the sum $a^j + (-1)a^i$, and the boundary of the face (a^i, a^j, a^k) as the sum of the edges that bound this face (with that circumnavigation direction which was fixed for the face), i.e., $(a^i, a^j) + (a^j, a^k) + (a^k, a^i)$; we put the boundary of a vertex equal to zero. The boundary operators thus defined are extended to the groups of chains by linearity. A chain whose boundary equals zero will be called a *cycle*; thus, a cycle is an algebraic analogue of a closed manifold (without boundary).

Since we are interested in homology groups, i.e., in classes of equivalent cycles which differ from one another by a boundary, we will consider the cosets of q-dimensional cycles with respect to the subgroup of the boundaries of $(q + 1)$-dimensional chains ($q = 0, 1, 2$). These cosets form a group called the q-dimensional homology group of the surface Π^2. The homology groups of Π^2 are easy to calculate; they are isomorphic to Z, 0 and Z for dimensions 0, 1 and 2, respectively. Such a construction may be performed for greater dimensions by using decompositions into tetrahedra and their analogues (simplexes). If we know how a given space is divided into simplexes, then we can compute its homology groups. In practice, however, the definition is rarely used to calculate homology groups. They employ various techniques to this end (exact sequences, spectral sequences, etc.).

* The direct sum $\bigoplus_{\alpha \in A} G_\alpha$ of groups $G_\alpha = \{g \cdot r_\alpha\}$, $g \in G$, isomorphic to G, where the isomorphism is given by the rule $g \cdot r_\alpha \to g$, is called the group of formal linear combinations of elements r_α from a certain set $\{r_\alpha\}_{\alpha \in A}$ with coefficients in an Abelian group G.

2. HOMOLOGY GROUPS OF CHAIN COMPLEXES

We begin with abstract algebraic objects.

A sequence (infinite)

$$\dots \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \dots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \tag{1}$$

of Abelian groups C_k and their homomorphisms ∂_k satisfying the condition $\partial_{k-1}\partial_k = 0$ for any $k \geqslant 1$ is called a *chain complex*. We will denote it by C_*; the groups C_k are called *chain groups* and the homomorphisms ∂_k are called *differentials* or *boundary homomorphisms*.

The set $\operatorname{Ker} \partial_k = \{c \in C_k : \partial_k c = 0\}$ forms a subgroup in C_k called a *group of k-dimensional cycles*; its elements are called *k-dimensional cycles*. The set $\operatorname{Im} \partial_{k+1} = \{c \in C_k : c = \partial_{k+1} u\}$ also forms a subgroup in C_k called a *group of k-dimensional boundaries*; its elements are called *k-dimensional boundaries*.

A sequence of homomorphisms $\varphi_k : C_k \to C_k'$ such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_k \arrow[r, "\partial_k"] \arrow[d, "\varphi_k"] & C_{k-1} \arrow[r] \arrow[d, "\varphi_{k-1}"] & \cdots \arrow[r] & C_1 \arrow[r] \arrow[d, "\varphi_1"] & C_0 \arrow[r] \arrow[d, "\varphi_0"] & 0 \\
\cdots \arrow[r] & C_k' \arrow[r, "\partial_k'"] & C_{k-1}' \arrow[r] & \cdots \arrow[r] & C_1' \arrow[r] & C_0' \arrow[r] & 0
\end{tikzcd} \tag{2}$$

is commutative, i.e., $\varphi_{k-1}\partial_k = \partial_k'\varphi_k$ for any k, is called a *homomorphism φ_* of a chain complex C_* to a chain complex C_*'*.

Let us introduce one of the most important notions of algebraic topology, viz., of homology group. Consider a chain complex C_*. Due to the relationship $\partial_k\partial_{k+1} = 0$, the inclusion relation $\operatorname{Im} \partial_{k+1} \subset \operatorname{Ker} \partial_k$ is fulfilled. The factor group of the group of cycles with respect to the group of boundaries $\operatorname{Ker} \partial_k / \operatorname{Im} \partial_{k+1}$ is called a *k-homology group of the complex C_** and denoted by $H_k(C_*)$. Cycles c_1, c_2 from one coset are said to be *homologous* and denoted by $c_1 \sim c_2$.

Let $\varphi_* : C_* \to C_*'$ be a homomorphism of chain complexes. It immediately follows from the commutativity of diagram (2) that

$$\varphi_k(\operatorname{Ker} \partial_k) \subset \operatorname{Ker} \partial_k' \text{ and } \varphi_k(\operatorname{Im} \partial_{k+1}) \subset \operatorname{Im} \partial'_{k+1}.$$

Therefore φ_* induces a *homomorphism of homology groups*:

$$\varphi_{*k} : H_k(C_*) \to H_k(C_*').$$

We continue the investigation of chain complexes and their homology groups. Let C_* and C_*^0 be chain complexes such that the groups C_k^0 are subgroups of the groups C_k, and the differentials ∂_k^0 of the complex C_*^0 are obtained by restricting ∂_k to C_k^0. In this case, the complex C_*^0 is called a *subcomplex of the complex C_**. A homomorphism $i_* : C_*^0 \to C_*$ of chain complexes is defined, where $i_k : C_k^0 \to C_k$ is an embedding monomorphism; i_* is called a *chain complex embedding monomorphism*.

Consider a sequence of factor groups $\hat{C}_k = C_k / C_k^0$. The homomorphisms ∂_k induce the homomorphisms $\hat{\partial}_k : \hat{C}_k \to \hat{C}_{k-1}$.

Exercise 1°. Show that the groups $\hat{C}_k$ and homomorphisms $\hat{\partial}_k$ form a chain complex $\hat{C}_*$, and the epimorphisms to factor groups $j_k : C_k \to \hat{C}_k$ form a homomorphism of chain complexes $j_* : C_* \to \hat{C}_*$ (an epimorphism to a quotient complex).

The sequence

$$\ldots \to A_{k+1} \xrightarrow{\psi_{k+1}} A_k \xrightarrow{\psi_k} A_{k-1} \to \ldots$$

of groups A_k and of their homomorphisms ψ_k is said to be *exact* if for every k, the image of the homomorphism ψ_{k+1} coincides with the kernel of the homomorphism ψ_k, i.e., $\operatorname{Im} \psi_{k+1} = \operatorname{Ker} \psi_k$.

Exercise 2°. Show that the sequence

$$0 \to C_k^0 \xrightarrow{i_k} C_k \xrightarrow{j_k} \hat{C}_k \to 0$$

is exact for every k.

The sequence of chain complexes and their homomorphisms

$$0 \to C_*^0 \xrightarrow{i_*} C_* \xrightarrow{j_*} \hat{C}_* \to 0, \tag{3}$$

where i_* is an embedding, j_* a factorization, is said to be exact.

According to the general definition, the homology groups of a quotient complex $\hat{C}_*$, i.e., the groups $H_k(\hat{C}_*)$, can be constructed. The new groups prove to be related to the groups $H_k(\hat{C}_*)$ and $H_k(C_*^0)$ by a certain exact sequence.

Let us construct this sequence. The homomorphisms i_* and j_* induce the homomorphisms

$$i_{*k} : H_k(C_*^0) \to H_k(C_*), \qquad j_{*k} : H_k(C_*) \to H_k(\hat{C}_*).$$

We obtain the short sequences

$$\begin{tikzcd}
H_k(C_*^0) \arrow[r, "i_{*k}"] & H_k(C_*) \arrow[r, "j_{*k}"] & H_k(\hat{C}_*) \arrow[dll, dashed, "\delta_k"] \\
H_{k-1}(C_*^0) \arrow[r, "i_{*k-1}"] & H_{k-1}(C_*) \arrow[r, "j_{*k-1}"] & H_{k-1}(\hat{C}_*)
\end{tikzcd}$$

There happen to exist homomorphisms

$$\delta_k : H_k(\hat{C}_*) \to H_{k-1}(C_*^0)$$

combining these short sequences into the long exact sequence

$$\ldots \to H_{k+1}(\hat{C}_*) \xrightarrow{\delta_{k+1}} H_k(C_*^0) \xrightarrow{i_{*k}} H_k(C_*) \xrightarrow{j_{*k}} H_k(\hat{C}_*)$$

$$\xrightarrow{\delta_k} H_{k-1}(C_*^0) \xrightarrow{i_{*k-1}} H_{k-1}(C_*) \to \ldots \to H_0(\hat{C}_*) \to 0. \tag{4}$$

To describe the construction of the homomorphisms δ_k, we consider sequence (3). Let $\tilde{\alpha} \in H_k(\hat{C}_*)$, $k > 0$, i.e., $\tilde{\alpha}$ is the coset of a certain element $\alpha \in \operatorname{Ker} \hat{\partial}_k$ respective to the subgroup $\operatorname{Im} \hat{\partial}_{k+1}$. In turn, $\alpha \in \hat{C}_k$ and can be considered as the coset of a certain element $d \in C_k$ respective to the subgroup C_k^0. It follows from $\hat{\partial}_k \alpha = 0$ that $\partial_k d \in C_{k-1}^0$ and that $\partial_k d \in \operatorname{Ker} \partial_{k-1} \subset C_{k-1}^0$ from $\partial_{k-1}\partial_k = 0$.

Exercise 3°. Show that the coset $[\partial_k d]^0$ of an element $\partial_k d$ in $H_{k-1}(C_*^0)$ does not depend on the choice of the elements α and d from the corresponding cosets.

We associated each element $\hat{\alpha}$ from $H_k(\hat{C}_*)$ with the element $[\partial_k d]^0$ from $H_{k-1}(C_*^0)$ thereby specifying a mapping which we will denote by

$$\delta_k : H_k(\hat{C}_*) \to H_{k-1}(C_*^0)$$

and call a *connecting homomorphism*.

Exercise 4°. Show that δ_k is, in fact, a homomorphism.

The construction of a connecting homomorphism may be extended by putting $\delta_0 : H_0(\hat{C}_*) \to 0$.

LEMMA 1. *Sequence* (4) *is exact*.

The proof is reduced to a direct check of relations

$$\operatorname{Im} \delta_{k+1} = \operatorname{Ker} i_{*k}, \quad \operatorname{Im} i_{*k} = \operatorname{Ker} j_{*k}, \quad \operatorname{Im} j_{*k} = \operatorname{Ker} \delta_k$$

and left to the reader.

3. HOMOLOGY GROUPS OF SIMPLICIAL COMPLEXES

Here, the algebraic technique developed in Sec. 2 is applied to the construction of homology groups of geometric objects.

1. Simplicial Complexes and Polyhedra. We first give necessary definitions.

DEFINITION 1. A *standard k-dimensional simplex* σ^k, $k \geqslant 0$, is the convex closure of $k + 1$ points in R^{k+1} with the coordinates $(1, 0, 0, \ldots, 0, 0)$, $(0, 1, 0, \ldots, 0, 0)$, $\ldots$, $(0, 0, \ldots, 0, 1)$, i.e., the collection of points with coordinates $(t_0, \ldots, t_k)$ such that $t_i \geqslant 0$ for each i and $\sum_{i=0}^{k} t_i = 1$.

DEFINITION 2. A *simplex of dimension* k or a *k-dimensional simplex* $\tau^k = (a^0, a^1, \ldots, a^k)$ is the convex closure of $k + 1$ points $a^0, \ldots, a^k$ of the Euclidean space R^n, $k \leqslant n$, lying in general position (not lying in the same m-plane of dimension less than k), i.e., the collection of points of the form $x = \sum_{i=0}^{k} t_i a^i$, where $t_i \geqslant 0$ for each i, $\sum_{i=0}^{k} t_i = 1$.

The points a^i are called the *vertices of the simplex* $(a^0, \ldots, a^k)$, and the numbers t_i the *barycentric coordinates* of the point $x \in (a^0, \ldots, a^k)$.

The notion of face of a simplex is defined in a natural way.

DEFINITION 3. The convex closure of a subset consisting of $s + 1$ vertices of the simplex τ^k, where $0 \leqslant s \leqslant k$, is called a *face of dimension s* or an *s-dimensional face of the k-dimensional simplex* τ^k. We will call faces of dimension $s < k$ of the simplex τ^k *proper*.

It is obvious that an s-dimensional face of a simplex is an s-dimensional simplex. In particular, the faces of a standard simplex (and the standard simplex itself) are simplexes. It is easy to verify that a k-dimensional simplex is affinely homeomorphic to a standard simplex of the same dimension; the interior (in the carrier k-plane) of a simplex τ^k can be considered as a special case of a k-dimensional cell

Thus, cell complexes can be constructed from simplexes of different dimensions. The fact that a simplex has faces enables us to connect simplexes in a more ordered manner than cells in the generic cell complex.

DEFINITION 4. A set $\{\tau_i^k\}$ of simplexes in R^n that satisfies the following conditions: (i) together with each k-dimensional simplex τ_i^k, any of its faces is included in K; (ii) two simplexes can intersect only in their common face, is called a *simplicial complex K*.

A simplicial complex is said to be *finite* if it consists of a finite number of simplexes.

Consider the set-theoretic union $|K| \subset R^n$ of all simplexes from K. Introduce on the set $|K|$ a topology that is the strongest of all those in which the embedding mapping of each simplex into $|K|$ is continuous. In other words, the set $A \subset |K|$ is closed if and only if $A \cap \tau_i^k$ is closed in τ_i^k for any $\tau_i^k \in K$. If the simplicial complex K is finite then this topology coincides with that induced by the metric on R^n.

DEFINITION 5. A space $|K|$ and, more generally, any topological space X homeomorphic to $|K|$ is called a *polyhedron*.

DEFINITION 6. Given a polyhedron X, a simplicial complex K such that the space $|K|$ is homeomorphic to X is called a *triangulation of the polyhedron X*.

Examples of polyhedra are the closed surfaces from Sec. 4, Ch. II. Their triangulation is given by partitioning a surface into topological triangles, their edges and vertices.

Consider a finite simplicial complex K. Fix in the space $|K| \subset R^n$ a metric from R^n. It is obvious that there exist different triangulations of the space $|K|$. Let K' be a triangulation of $|K|$. The greatest of the lengths of 1-dimensional simplexes included in K' is called the *fineness of the triangulation K'*.

Exercises.

1°. Prove that a polyhedron is (a) a normal Hausdorff space; (b) a cell complex.
2° Prove that if K is a finite simplicial complex, then the space $|K|$ is (a) a compact space; (b) a finite cell complex.
3°. Prove that a simplicial complex K is finite if and only if the polyhedron $|K|$ is compact.

Below, if it is not stated otherwise, we will consider finite simplicial complexes and compact polyhedra. It is easy to see that a compact polyhedron is a metrizable space.

Let X be a polyhedron, K a simplicial complex and $\varphi : |K| \to X$ a homeomorphism. The homeomorphism φ generates a decomposition (triangulation) of the space X into the sets $\Sigma_i^k = \varphi(\tau_i^k)$, $\tau_i^k \in K$, which are called *curvilinear simplexes*; the images of the vertices of the simplex τ_i^k are called the *vertices of the curvilinear simplex* Σ_i^k.

Exercises.
4°. Show that the closed disc $\overline{D}^n$ and sphere S^{n-1} are polyhedra and specify their decomposition into curvilinear simplexes.
5°. Show that (a) the set $\{\tau^n\}$, the collection of the simplex τ^n and all its faces, $|\{\tau^n\}| = \tau^n$, and (b) the set $\{\partial\tau^n\}$, the collection of proper faces of the simplex τ^n, where $|\{\partial\tau^n\}|$ coincides with the boundary $\partial\tau^n$ of the set τ^n in the carrier n-plane, are simplicial complexes.

2. Homology Groups of Simplicial Complexes and Polyhedra.

Now, we associate a simplicial complex K with some chain complex and enumerate the vertices of each simplex $\tau_i^k \in K$ by the numbers $0, 1, \dots, k$ in some order a^{i_0}, $a^{i_1}, \dots, a^{i_k}$. There are $(k+1)!$ such numerations. Two numerations are said to be *equivalent* if one of them can be obtained from the other by transposing the numbers an even number of times. The set of all numerations is thus decomposed into two equivalence classes denoted by Λ_i^+ and Λ_i^-, respectively.

DEFINITION 7. A simplex τ^k with one of the classes Λ^+, Λ^- being indicated, i.e., one of the pairs (τ^k, Λ^+), (τ^k, Λ^-), is called an *oriented simplex*, and the corresponding class its *orientation*.

It is more convenient to write an oriented simplex (τ_i^k, Λ_i^+) in a different way, viz., by specifying some numeration $a^{i_0}, a^{i_1}, \dots, a^{i_k}$ from the orientation class and denoting it as follows:

$$(\tau_i^k, \Lambda_i^+) = [a^{i_0}, a^{i_1}, a^{i_2}, \dots, a^{i_k}];$$

then

$$(\tau_i^k, \Lambda_i^-) = [a^{i_1}, a^{i_0}, a^{i_2}, \dots, a^{i_k}].$$

DEFINITION 8. The factor group of the group of formal linear combinations (finite) of the form $\sum_i g_i \cdot (\tau_i^k, \Lambda_i)$, where $g_i \in G$, $\Lambda_i = \Lambda_i^+$ or $\Lambda_i = \Lambda_i^-$, respective to the subgroup of elements of the form

$$g \cdot (\tau_i^k, \Lambda_i^+) + g \cdot (\tau_i^k, \Lambda_i^-) \tag{1}$$

and their linear combinations is called the *group of k-dimensional chains* $C_k(K; G)$ of the simplicial complex K with coefficients in the Abelian group G.

In other words, we identify the elements $g \cdot (\tau_i^k, \Lambda_i^-)$, $-g \cdot (\tau_i^k, \Lambda_i^+)$ in the group of formal linear combinations of oriented simplexes.

The differential

$$\partial_k : C_k(K; G) \to C_{k-1}(K; G)$$

is defined by the equality

$$\partial_k(g \cdot [a^{i_0}, a^{i_1}, \dots, a^{i_k}]) = \sum_{j=0}^{k} (-1)^j g \cdot [a^{i_0}, \dots, a^{i_{j-1}}, a^{i_{j+1}}, \dots, a^{i_k}] \tag{2}$$

for each oriented simplex. We extend it to the whole group $C_k(K; G)$ by additivity.

For $k = 0$, we put $\partial_0 : C_0(K; G) \to 0$.

PROPOSITION 1. *For all $k \geqslant 1$, the equality $\partial_{k-1}\partial_k = 0$ holds.*

PROOF. In fact, in the sum $\partial_{k-1}\partial_k(g \cdot [a^{i_0}, \dots, a^{i_k}])$, there are simultaneously the following addends

$$(-1)^p(-1)^{q-1} g \cdot [a^{i_0}, \dots, a^{i_{p-1}}, a^{i_{p+1}}, \dots, a^{i_{q-1}}, a^{i_{q+1}}, \dots, a^{i_k}]$$

and

$$(-1)^p(-1)^q g \cdot [a^{i_0}, \dots, a^{i_{p-1}}, a^{i_{p+1}}, \dots, a^{i_{q-1}}, a^{i_{q+1}}, \dots, a^{i_k}]$$

which eliminate each other. ■

Thus, the groups $C_k(K; G)$ and differentials ∂_k form a chain complex denoted by $C_*(K; G)$. E.g., the group of integers Z may be taken as G.

DEFINITION 9. The homology groups of a chain complex $C_*(K; G)$ are called the *homology groups of the simplicial complex K* with coefficients in the Abelian group G and denoted by $H_k(K; G)$.

DEFINITION 10. The homology groups of a triangulation K of a polyhedron X with coefficients in an Abelian group G are called the *homology groups $H_k(X; G)$ of the polyhedron X.*

The correctness of this definition (i.e., independence from the choice of a triangulation) is proved by a complicated technique; we will discuss these topics in Sec. 5.

3. Calculation of Homology Groups of Concrete Polyhedra.

Let us calculate the homology groups $H_k(\tau^n; G)$ of a polyhedron τ^n. It is obvious that for τ^0, i.e., a space consisting of one point, we have

$$C_k(\{\tau^0\}; G) = \operatorname{Ker} \partial_k = \operatorname{Im} \partial_k = 0 \quad \text{when} \quad k > 0; \quad C_0(\{\tau^0\}; G) = \operatorname{Ker} \partial_0 = G.$$

Hence, we obtain the homology groups

$$H_k(\tau^0; G) = 0 \quad \text{when} \quad k > 0; \quad H_0(\tau^0; G) = G. \tag{3}$$

Before calculating $H_k(\tau^n; G)$ when $n > 0$, we solve a more general problem. Consider a simplicial complex K lying in the hyperplane $\Pi^m \subset R^{m+1}$ and a point $a \in R^{m+1} \setminus \Pi^m$. We will call the collection of simplexes consisting of simplexes $\tau_i^k \in K$, the simplex a and simplexes of the form (a, τ_i^k), i.e., simplexes $(a, a^{i_0}, \dots, a^{i_k})$ such that $\tau_i^k = (a^{i_0}, \dots, a^{i_k})$ is a certain simplex in K, the *cone aK over the complex K with the vertex a.*

Exercise 6°. Show that aK is a simplicial complex.

PROPOSITION 2. Let aK be a cone with a vertex a over a simplicial complex K. Then

$$H_k(aK;G) = 0 \quad \text{when} \quad k > 0; \quad H_0(aK;G) = G \tag{4}$$

PROOF. Consider an arbitrary 0-dimensional chain $g \cdot a + \sum_i g_i \cdot a^i$ from $C_0(aK; G) = \operatorname{Ker} \partial_0$; we have

$$g\, a + \sum_i g_i \cdot a^i = \Big(g + \sum_i g_i\Big) \cdot a + \sum_i (g_i \cdot a^i - g_i a).$$

Due to the equality

$$\sum_i (g_i \cdot a^i - g_i a) = \partial_1 \Big(\sum_i g_i [a, a^i]\Big),$$

an arbitrary cycle $g \cdot a + \sum_i g_i \cdot a^i$ from $\operatorname{Ker} \partial_0$ is homologous to the cycle $g' \cdot a = \Big(g + \sum_i g_i\Big) \cdot a$ which is not homologous to zero in the group $C_0(aK; G)$ when $g' \neq 0$. We obtain the isomorphism $H_0(aK; G) = G$.

Consider now an arbitrary k-dimensional cycle in $C_k(aK; G)$

$$z_k = \sum_i g_i \cdot [\tau_i^k] + \sum_j h_j \cdot [a, \tau_j^{k-1}] \in \operatorname{Ker} \partial_k,$$

where $i \in I_k$, $j \in I_{k-1}$, $g_i, h_i \in G$ and $[\tau_i^k]$, $[a, \tau_j^{k-1}]$ den ce oriented simplexes.

We have

$$\sum_i g_i \cdot [\tau_i^k] - \sum_i (g_i \cdot [\tau_i^k] - \partial_{k+1}(g_i \cdot [a, \tau_i^k])) = \sum_j g_j' \cdot [a, \tau_j^{k-1}].$$

Therefore the cycle z_k is homologous to the cycle

$$z_k' = \sum_j h_j' \cdot [a, \tau_j^{k-1}] = \sum_j (g_j' + h_j) \cdot [a, \tau_j^{k-1}].$$

The coefficient of the simplex $[\tau_j^{k-1}]$ in the sum $\partial_k \Big(\sum_j h_j' \cdot [a, \tau_j^{k-1}]\Big)$ is h_j' (there being only one such simplex!). Therefore $\sum_j h_j' \cdot [a, \tau_j^{k-1}]$ is a cycle if and only if $h_j' = 0$ for each j.

Thus, we have established that in $C_*(aK; G)$, when $k > 0$, any cycle from $\operatorname{Ker} \partial_k$ is homologous to zero in $C_k(aK; G)$. Therefore, $H_k(aK, G) = 0$ when $k > 0$. ■

Note that the complex $[\tau^n]$ corresponding to the simplex $\tau^n = (a^0, \ldots, a^n)$ is a cone $a^0[\tau^{n-1}]$ with the vertex a^0 over the complex $[\tau^{n-1}]$ which corresponds to the simplex $\tau^{n-1} = (a^1, \ldots, a^n)$. Therefore, from equalities (3) and (4), we obtain the homology groups of an n-dimensional simplex:

$$H_k(\tau^n; G) = \begin{cases} 0 & \text{when} \quad k > 0, \\ G & \text{when} \quad k = 0 \end{cases} \tag{5}$$

for each $n \geqslant 0$.

We now calculate the homology groups $H_k(|\{\partial\tau^n\}|; G)$ of a polyhedron $|\{\partial\tau^n\}|$ whose triangulation $\{\partial\tau^n\}$ consists of all proper faces of the simplex τ^n. Consider the case when $n > 1$. When $k < n$, we have

$$C_k(\{\partial\tau^n\}; G) = C_k(\{\tau^n\}; G),$$

and the differentials of the chain complexes $C_*(\{\partial\tau^n\}; G)$ and $C_*(\{\tau^n\}; G)$ coincide. Therefore, when $k < n - 1$,

$$H_k(\{\partial\tau^n\}; G) = H_k(\{\tau^n\}; G). \tag{6}$$

It is obvious that when $k > n - 1$

$$H_k(\{\partial\tau^n\}; G) = 0. \tag{7}$$

Since $H_{n-1}(\{\tau^n\}; G) = 0$, any cycle $z^{n-1} \in C_{n-1}(\{\tau^n\}; G)$ is the boundary $\partial_n(g \cdot [\tau^n])$ of the chain $g \cdot [\tau^n] \in C_n(\{\tau^n\}; G)$, and therefore in the complex $C_*(\{\tau^n\}; G)$, we have $\operatorname{Ker} \partial_{n-1} = \operatorname{Im} \partial_n \cong G$. The differentials in the complexes $C_*(\{\tau^n\}; G)$ and $C_*(\{\partial\tau^n\}; G)$ coincide on the chain groups $C_{n-1}(\{\tau^n\}; G) = C_{n-1}(\{\partial\tau^n\}; G)$. Therefore, in $C_*(\{\partial\tau^n\}; G)$, the group $\operatorname{Ker} \partial_{n-1}$ is isomorphic to the group G, whereas $\operatorname{Im} \partial_n = \partial_n(C_n(\{\partial\tau^n\}; G)) = 0$; therefore

$$H_{n-1}(\{\partial\tau^n\}; G) \cong G. \tag{8}$$

Thus, when $n > 1$, the homology groups of the boundary of an n-dimensional simplex have been calculated:

$$H_k(|\{\partial\tau^n\}|; G) = \begin{cases} 0 & \text{when} \quad k \neq 0, \ n - 1, \\ G & \text{when} \quad k = 0, \ n - 1. \end{cases} \tag{9}$$

Exercise 7°. Prove that

$$H_k(|\{\partial\tau^1\}|; G) = \begin{cases} 0 & \text{when} \quad k > 0, \\ G \oplus G & \text{when} \quad k = 0. \end{cases} \tag{10}$$

We now dwell on a geometric interpretation of the homology groups of a simplicial complex. A cycle from $C_k(K; Z)$ is a set of k-dimensional simplexes from K each of which is taken a certain number of times; this set is closed in the sense that each $(k - 1)$-dimensional simplex is included in the boundary of the k-dimensional cycle the same number of times with two opposite orientations. Two k-dimensional cycles are equivalent (homologous) if their difference is the boundary of a $(k + 1)$-dimensional chain, i.e., bounds a certain set of $(k + 1)$-dimensional simplexes; the group $H_k(|K|; Z)$ is the group of equivalence classes of such k-dimensional cycles. Roughly speaking, $H_k(|K|; Z)$ consists of those closed collections of k-dimensional simplexes which cannot be 'glued up' with collections of $(k + 1)$-dimensional simplexes. Thus, intuitively, the group $H_k(|K|; Z)$ corresponds to the group generated by $(k + 1)$-dimensional 'openings' in the space $|K|$.

DEFINITION 11. A *subcomplex of a simplicial complex* is a subset L (which is a simplicial complex) of simplexes from K.

Let L be a subcomplex of a simplicial complex K. It is obvious that $C_*(L; G)$ is a subcomplex of the chain complex $C_*(K; G)$. Therefore a quotient complex is defined

$$C_*(K, L; G) = C_*(K; G)/(C_*(L; G).$$

Denoting the homology groups of this chain complex by $H_k(K, L; G)$, from the exact sequence of chain complexes

$$0 \to C_*(L; G) \xrightarrow{i_*} C_*(K; G) \xrightarrow{j_*} C_*(K, L; G) \to 0,$$

we obtain a long exact sequence of homology groups

$$\ldots \to H_{k+1}(K, L; G) \xrightarrow{\delta_{k+1}} H_k(L; G) \xrightarrow{i_{*k}} H_k(K; G)$$

$$\xrightarrow{j_{*k}} H_k(K, L; G) \xrightarrow{\delta_k} H_{k-1}(L; G) \to \ldots .$$

It is called an *exact sequence of the pair* (K, L), the groups $H_k(K, L; G)$ are called *relative homology groups or homology groups of the pair* (K, L).

It will be useful to 'decode' the definition of relative homology groups.

Since the chain $\hat{\gamma}_k$ from $C_k(K, L; G)$ is a coset of the group $C_k(K; G)$ relative to the subgroup $i_kC_k(L; G) = C_k(L, G)$, in the coset $\hat{\gamma}_k$, there exists a unique representative, the chain γ_k from $C_k(K; G)$, which includes only those oriented simplexes with nonzero coefficients of the complex K that are not oriented simplexes of the subcomplex L. It follows from the definition of a boundary homomorphism in a quotient complex that the boundary homomorphism $\hat{\partial}_k : C_k(K, L; G) \to C_{k-1}(K, L; G)$ transforms the chain $\hat{\gamma}_k$ into a chain $\hat{\gamma}_{k-1}$ which is the coset of the group $C_{k-1}(K; G)$ relative to the subgroup $i_{k-1}C_{k-1}(L; G) = C_{k-1}(L; G)$ with the representative $\partial_k\gamma_k \in C_{k-1}(K; G)$. In the chain $\partial_k\gamma_k$, we discard all addends $g_m[\tau_m^{k-1}]$ for which τ_m^{k-1} is a simplex from L. Obviously, the obtained chain γ_{k-1} belongs to the same coset $\hat{\gamma}_{k-1}$ as the chain $\partial_k\gamma_k$.

It is clear that a chain complex $C_*(K, L; G)$ is isomorphic to the chain complex $\bar{C}_*$ whose chains are formal linear combinations of oriented simplexes (in the sense of Definition 8) from $K \setminus L$, and the boundary homomorphism associates the k-dimensional chain γ_k with a chain of dimension $k - 1$ obtained by calculating on γ_k the value of the boundary homomorphism ∂_k (in the chain complex $C_*(K; G)$) and deleting all extraneous addends, i.e., those addends $g_m[\tau_m^{k-1}]$ for which τ_m^{k-1} belongs to L. Since an isomorphism of chain complexes induces an isomorphism of homology groups,

$$H_k(\bar{C}_*) = H_k(K, L; G), \quad k = 0, 1, 2, \ldots .$$

Thus, we have arrived at a more geometry-oriented definition of the homology groups of a pair. Note that the chains, cycles and boundaries of the complex $\bar{C}_*$ are said to be relative (for the pair (K, L)).

Now, we make out the geometric meaning of the connecting homomorphism

$$\delta_k : H_k(K, L; G) \to H_{k-1}(L; G).$$

Let $\bar{h}_k \in H_k(K, L; G)$ be a homology class of the relative cycle $z_k \in \check{C}_k$. Consider z_k as a chain in $C_*(K; G)$ and calculate its boundary $\partial_k z_k$ in it. By the definition of a relative cycle, after collecting like terms, the chain $\partial_k z_k$ will include with nonzero coefficients only oriented simplexes from L. Therefore $\partial_k z_k$ can be considered as a chain in $C_*(L; G)$. It is verified quite simply that $\partial_k z_k$ is a cycle whose homology class $h_{k-1} \in H_{k-1}(L; G)$ does not depend on the choice of the representative z_k of the class $\bar{h}_k$. According to the general structure of the connecting homomorphism (Sec. 2), $\delta_k \bar{h}_k = h_{k-1}$. If we imagine a relative cycle as a manifold with boundary lying in L and made up of k-dimensional oriented simplexes, then $\partial_k z_k$ is just this boundary with the corresponding orientations of $(k-1)$-dimensional simplexes.

EXAMPLE (see Fig. 109). Let a simplicial complex k consist of simplexes

$$a^0, a^1, a^2, a^3,$$
$$(a^0, a^1), (a^1, a^2), (a^2, a^3), (a^3, a^0), (a^1, a^3)$$
$$(a^0, a^1, a^3), (a^1, a^2, a^3),$$

and let its subcomplex L consist of the same simplexes except

$$(a^1, a^2), (a^0, a^1, a^3), (a^1, a^2, a^3).$$

Thus, $|K|$ is a rectangle (with the 'interior'), and $|L|$ its boundary. It is obvious that the chain $\gamma_2 \in C_2(K, Z)$, $\gamma_2 = [a^0, a^1, a^3] + [a^1, a^2, a^3]$ is a relative cycle of the pair (K, L). In fact, its boundary $\partial_2\gamma_2 = [a^3, a^0] + [a^0, a^1] + [a^1, a^2] + [a^2, a^3]$ includes with nonzero coefficients only oriented simplexes from the subcomplex L. The chain $\gamma_1 [a^1, a^3]$ from $C_1(K; Z)$ is simultaneously a relative cycle (verify!) and a relative boundary, because it can be obtained from

$$\partial_2[a^0, a^1, a^3] = [a^1, a^3] + [a^3, a^0] + [a^0, a^1]$$

by discarding the addends $[a^3, a^0]$ and $[a^0, a^1]$ which are oriented simplexes from

Fig. 109

the subcomplex L. It is easy to see that the relative cycle γ_2 determines the generator of the group $H_2(K, L; Z) \simeq Z$. The connecting homomorphism $\delta_2 : H_2(K, L; Z) \to H_1(L; Z)$ associates this generator with an element (also generating) of the group $H_1(L; Z)$ which consists of one cycle $\partial_2\gamma_2$.

Exercises.

8°. Write the exact sequence of the pair (K, L) for the example considered.

9°. Let L_1 and L_2 be subcomplexes of a simplicial complex K. Prove that $L_1 \cap L_2$ and $L_1 \cup L_2$ are also subcomplexes of the complex K, and show that the sequence

$$0 \to C_*(L_1 \cap L_2; G) \xrightarrow{I_*} C_*(L_1; G) \oplus C_*(L_2; G) \to C_*(L_1 \cup L_2; G) \to 0,$$

$$\text{where } I_k\Big(\sum_i g_i \cdot [\tau_i^k]\Big) = \Big(\sum_i g_i \cdot [\tau_i^k], -\sum_i g_i \cdot [\tau_i^k]\Big),$$

is exact. Hence, derive the exact sequence

$$\ldots \to H_{k+1}(L_1 \cup L_2; G) \to H_k(L_1 \cap L_2; G) \to H_k(L_1; G)$$
$$\oplus H_k(L_2; G) \to H_k(L_1 \cup L_2; G) \to H_{k-1}(L_1 \cap L_2; G) \to \ldots \quad (11)$$

called the *Mayer-Vietoris exact sequence.*

Exact sequence (11) enables us to calculate the homology groups of complicated simplicial complexes.

10°. Using (5), (9), (10) and (11), calculate the homology groups of the complex consisting of simplexes of dimensions 0 and 1, and drawn in Fig. 110.

Hint: Consider the complex as a consequent union of subcomplexes.

11°. Show that for an orientable surface M_p of genus p, we have the isomorphism $H_2(M_p; Z) \simeq Z$.

Hint: Show that any two-dimensional cycle is a multiple of a cycle which is equal to the sum of all curvilinear 2-simplexes of a triangulation of M_p which are taken with a compatible orientation.

12°. Show that for an oriented surface M_p of genus p, we have

$$H_1(M_p; Z) = \underbrace{Z \oplus \ldots \oplus Z}_{2p}.$$

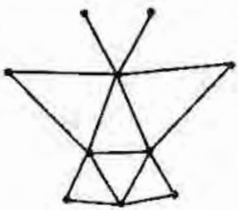


Fig. 110

Hint: Use the Mayer-Vietoris exact sequence.

13°. Show that

$$H_k(RP^2; Z) = \begin{cases} Z, & k = 0, \\ Z_2, & k = 1, \\ 0, & k > 1 \end{cases}$$

and

$$H_k(RP^2; Z_2) = \begin{cases} Z_2, & k = 0, 1, 2, \\ 0, & k > 2. \end{cases}$$

Hint: Use a simplicial partition of RP^2.

14°. Show that for a non-orientable surface N_q of genus q

$$H_k(N_q; Z) = \begin{cases} Z, & k = 0, \\ \underbrace{Z \oplus \dots \oplus Z}_{q-1} \oplus Z_2, & k = 1, \\ 0, & k > 1. \end{cases}$$

15°. Show that

$$H_k(M_p; Z_2) = \begin{cases} Z_2, & k = 0, 2, \\ \underbrace{Z_2 \oplus \dots \oplus Z_2}_{2p}, & k = 1, \\ 0, & k > 2 \end{cases}$$

and

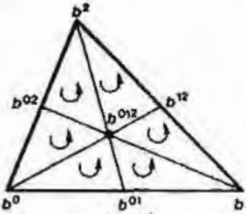
$$H_k(N_q; Z_2) = \begin{cases} Z_2, & k = 0, 2, \\ \underbrace{Z_2 \oplus \dots \oplus Z_2}_{q}, & k = 1, \\ 0, & k > 2. \end{cases}$$

4. Barycentric Subdivisions. Simplicial Mappings.

Let $\tau^k = (a^0, \dots, a^k)$ be a k-dimensional simplex. A point with barycentric coordinates $1/(k+1), \dots, 1/(k+1)$ is called the *barycentre of the simplex* τ^k. Denote this point by $b^{0, 1, \dots, k}$; more generally, denote by $b^{i_0, \dots, i_p}$ a point whose barycentric coordinates t_i are defined as follows

$$t_i = \begin{cases} \dfrac{1}{p+1}, & i = i_0, \dots, i_p, \\ 0 & \text{otherwise.} \end{cases}$$

For all possible sets $a^{i_0}, \dots, a^{i_p}$ of $p + 1$ vertices $(0 \leqslant p \leqslant k)$, the points $b^{i_0, \dots, i_p}$ corresponding to them are the barycentres of the p-dimensional faces $(a^{i_0}, \dots, a^{i_p})$ of



$\left\{ \tau^2 \right\}'$

Fig. 111

the simplex τ^k (remember that 0-dimensional faces are the vertices a^i, and the k-dimensional face the simplex τ^k itself). Consider all possible simplexes of the form

$$(b^{i_0, i_1, \dots, i_{p-1}, i_p}, b^{i_0, i_1, \dots, i_{p-1}}, \dots, b^{i_0, i_1}, b^{i_0}), \qquad 0 \leqslant p \leqslant k.$$

The collection of all such simplexes and their faces forms a simplicial complex called a *barycentric subdivision of the simplex* τ^k (Fig. 111).

Let K be a simplicial complex. Barycentric subdivisions of all its simplexes form a simplicial complex K' called a *barycentric subdivision of the complex* K. We will also consider simplicial complexes $K^{(2)} = (K')', \dots, K^{(r)} = (K^{(r-1)})'$.

The operation of subdividing the complex K barycentrically defines a chain homomorphism

$$\Theta_* : C_*(K; G) \to C_*(K'; G).$$

The homomorphism Θ_0 is defined on the vertices a^i by the formula

$$\Theta_0(g \cdot a^i) = g \cdot a^i, \tag{12}$$

and on simplexes of greater dimension, it can be defined inductively by the formal relation

$$\Theta_p(g \cdot [a^{i_0}, \dots, a^{i_p}]) = [b^{i_0 \cdots i_p}, \Theta_{p-1}\partial_p(g \cdot [a^{i_0}, \dots, a^{i_p}])] \tag{13}$$

which means that if the equality

$$\Theta_{p-1}\partial_p(g \cdot [a^{i_0}, \dots, a^{i_p}]) = \sum_k g_k \cdot [c_k^{j_0}, \dots, c_k^{j_{p-1}}]$$

is held, then we have

$$\Theta_p(g \cdot [a^{i_0}, \dots, a^{i_p}]) = \sum_k g_k \cdot [b^{i_0 \cdots i_p}, c_k^{j_0}, \dots, c_k^{j_{p-1}}],$$

and Θ_p can be extended to the whole group $C_p(K; G)$ by linearity. It is easy to see that Θ_* is a chain homomorphism.

Together with $\Theta_\bullet : C_\bullet(K; G) \to C_\bullet(K'; G)$, the following homomorphisms are naturally defined:

$$\Theta_\bullet^{(r)} : C_\bullet(K; G) \to C_\bullet(K^{(r)}; G).$$

Let K and L be simplicial complexes. A mapping $f : |K| \to |L|$ is said to be *simplicial* if the image of each simplex τ^k from K is a certain simplex from L, and the mapping $f|_{\tau^k}$ is linear in barycentric coordinates:

$$f(t_0 a^{i_0} + \ldots + t_k a^{i_k}) = t_0 f(a^{i_0}) + \ldots + t_k f(a^{i_k}).$$

The notions of barycentric subdivision and simplicial mapping have meaning also in considering polyhedra made up of curvilinear simplexes because barycentric coordinates may be transferred to curvilinear simplexes by means of a triangulation homeomorphism.

Let $f : |K| \to |L|$ be a simplicial mapping. We define the homomorphisms $\hat{f}_p : C_p(K; G) \to C_p(L; G)$ as follows: for each simplex $(a^{i_0}, \ldots, a^{i_p}) \in K$, we put

$$\hat{f}_p(g \cdot [a^{i_0}, \ldots, a^{i_p}]) = \begin{cases} g \cdot [fa^{i_0}, \ldots, fa^{i_p}] & \text{if } (fa^{i_0}, \ldots, fa^{i_p}) \text{ is a simplex of dimension } p \\ 0 & \text{if } (fa^{i_0}, \ldots, fa^{i_p}) \text{ is a simplex of dimension less than } p \end{cases}$$

and extend $\hat{f}_p$ to $C_p(K; G)$ by linearity.

Exercises.

16°. Show that the collection of homomorphisms $\{\hat{f}_p\}$ is a chain complex homomorphism

$$\hat{f}_\bullet : C_\bullet(K; G) \to C_\bullet(L; G)$$

and therefore induces homomorphisms

$$f_{*p} : H_p(K; G) \to H_p(L; G).$$

17°. Show that simplicial mappings are morphisms of the category whose objects are simplicial complexes, and the correspondence

$$K \dashrightarrow H_p(K; G),$$
$$f : K \to L \dashrightarrow f_{*p} : H_p(K; G) \to H_p(L; G)$$

is a covariant functor from the above category to the category of Abelian groups.

18°. Show that the correspondence associating an Abelian group G with the homology group $H_k(K; G)$ of a simplicial complex K with coefficients in G is a covariant functor from the category of Abelian groups to the same category.

4. SINGULAR HOMOLOGY THEORY

1. Singular Homology Groups. In this section, another functor from the category of homotopy types of spaces to the category of Abelian groups, i.e., the

homology functor, will be constructed. To involve algebraic constructions of Sec. 2 for the purpose of studying the topological space, it is necessary to work out methods of constructing chain complexes from a given space X. In algebraic topology, there are a number of such techniques which assume the fulfilment of some or other properties for the space X; we give here one of the most general.

A continuous mapping $f^k : \sigma^k \to X$ of the standard simplex σ^k to a topological space X is called a *singular k-dimensional simplex* of the topological space X.

Let G be a ring with identity *, e.g., the ring Z of integers. A formal linear combination $\sum\limits_i g_i \cdot f_i^k$ of singular k-dimensional simplexes of the space X with the coefficients g_i from G, only a finite number of which differ from zero, is called a *k-dimensional singular chain* of the space X. The set of all k-dimensional singular chains of X with coefficients in G is denoted by $C_k^s(X; G)$. It is an Abelian group under the operation of addition of chains as linear combinations. If $G = Z$ then the group $C_k^s(X; Z)$ is free Abelian and its generators are all possible singular k-dimensional simplexes.

We define the differential

$$\partial_k^s : C_k^s(X; G) \to C_{k-1}^s(X; G).$$

To this end, consider the standard $(k-1)$- and k-dimensional simplexes σ^{k-1} and σ^k. Let us associate a point

$$(t_0, \ldots, t_{i-1}, t_i, \ldots, t_{k-1}) \in \sigma^{k-1}$$

with the point

$$(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{k-1}) \in \sigma^k.$$

This correspondence defines a mapping $\Delta_i^{k-1} : \sigma^{k-1} \to \sigma^k$ from σ^{k-1} onto the i-th $(k-1)$-dimensional face of the simplex σ^k. If f^k is a k-dimensional singular simplex then the superposition $f^k \Delta_i^{k-1}$ is evidently a $(k-1)$-dimensional singular simplex. For any simplex f^k, $k \geqslant 1$, we put

$$\partial_k^s f^k = \sum_{i=0}^{k} (-1)^i \cdot (f^k \Delta_i^{k-1}),$$

and define the homomorphism ∂_k^s on the whole group $C_k^s(X; G)$ by linearity:

$$\partial_k^s \Big(\sum_i g_i \cdot f_i^k \Big) = \sum_i g_i \cdot \partial_k^s f_i^k.$$

If $k = 0$ then it is natural to put $\partial_0^s f^0 = 0$ and, in accordance with the previous, to extend ∂_0^s by the zero value to $C_0^s(X; G)$.

---

* This has been done only with the simplification of notation in mind. All the constructions of this chapter can be performed for an arbitrary Abelian coefficient group G, just like in the previous section.

Exercise 1°. Verify that $\partial_k^s \partial_{k+1}^s = 0$.

Hint: It suffices to verify this equality on an arbitrary simplex f^{k+1}.

As we see, the sequence of groups $C_k^s(X; G)$ and homomorphisms ∂_k^s forms a chain complex which we denote by $C_*^s(X; G)$. It is called a *singular chain complex* of the space X.

Let $\varphi : X \to Y$ be a continuous mapping. For any k-dimensional singular simplex $f^k : \sigma^k \to X$ of the space X, the superposition φf^k is a k-dimensional singular simplex of the space Y. It is obvious that φ induces the homomorphism $\varphi_k : C_k^s(X; G) \to C_k^s(Y; G)$.

Exercise 2°. Prove that the system of homomorphisms φ_k forms a chain complex homomorphism

$$\varphi_* : C_*^s(X; G) \to C_*^s(Y; G),$$

i.e., for $k \geqslant 1$, the equalities hold: $\bar{\partial}_k^s \varphi_k = \varphi_{k-1} \partial_k^s$, where ∂_k^s, $\bar{\partial}_k^s$ are the differentials of the complexes $C_*^s(X; G)$, $C_*^s(Y; G)$.

DEFINITION 1. The homology groups of the complex $C_*^s(X; G)$ are called the *singular homology groups* of the space X with coefficients in G; a k-homology group is denoted by $H_k^s(X; G)$, and the collection of groups $\{H_k^s(X; G)\}_{k \geqslant 0}$ by $H_*^s(X; G)$.

EXAMPLE. Calculate the homology groups of the point $*$. It is obvious that $C_k^s(*; G) = G$ because there is only one singular simplex $f^k : \sigma^k \to *$ for any k. The value of the differential on it when $k \geqslant 1$ is calculated by the formula

$$\partial_k^s f^k = \sum_{i=0}^{k} (-1)^i \cdot f^k \Delta_i^{k-1} = \sum_{i=0}^{k} (-1)^i \cdot f^{k-1} = \begin{cases} 0 & \text{when } k \text{ is odd,} \\ f^{k-1} & \text{when } k \text{ is even.} \end{cases}$$

Remember that $\partial_k^s = 0$ when $k = 0$. Hence, we obtain that if k is odd, then

$$\operatorname{Im} \partial_{k+1}^s = C_k^s(*, G) = \operatorname{Ker} \partial_k^s = G;$$

if, however, k is even and not equal to zero, then

$$\operatorname{Im} \partial_{k+1}^s = \operatorname{Ker} \partial_k^s = 0.$$

Finally, $\operatorname{Im} \partial_1^s = 0$, $\operatorname{Ker} \partial_0^s = G$; therefore,

$$H_0^s(*; G) = G; \quad H_i^s(*; G) = 0, \quad i > 0. \; \blacklozenge \tag{1}$$

Since the continuous mapping $\varphi : X \to Y$ induces the homomorphism $\varphi_* : C_*^s(X; G) \to C_*^s(Y; G)$ of singular chain complexes of the spaces X and Y, it induces the *homomorphisms of singular homology groups*

$$\varphi_{*k} : H_k^s(X; G) \to H_k^s(Y; G).$$

Exercises.

3°. Show that if $\varphi : X \to Y$, $\psi : Y \to Z$ are continuous mappings, then $(\psi\varphi)_{*k} = \psi_{*k}\varphi_{*k}$. Show that to the identity mapping of X, there corresponds the

homology group identity mapping, i.e., $(1_X)_{*k} = 1_{H^s_k(X;G)}$. Hence, derive that the homology groups of homeomorphic spaces coincide (the homology group topological invariance theorem).

4°. Show that a constant mapping $X \to Y$, i.e., a mapping sending X to a point $y_0 \in Y$ induces the trivial (zero) homomorphism in homology groups of higher dimensions, $k > 0$.

2. Properties of Singular Homology Groups.

In Item 1, one covariant-functor, or more precisely, the collection of functors $H^s_* = \{H^s_k(*\,; G)\}_{k \geqslant 0}$ from the category of topological spaces to the category of Abelian groups was constructed. Let us study the most important properties of this functor.

THEOREM 1. *Let mappings $\varphi; \psi : X \to Y$ be homotopic. Then the induced homology group homomorphisms coincide.*

First, we prove the following statement.

LEMMA 1. *Let B be a convex set of a Euclidean space; then*

$$H^s_*(B; G) = H^s_*(*\,; G). \tag{2}$$

PROOF. Let $f^k : \sigma^k \to B$ be a singular simplex. We define the singular simplex $D_k f^k : \sigma^{k+1} \to B$ by the equality

$$D_k f^k(t_0, \ldots, t_{k+1}) = \begin{cases} t_0 w + (1 - t_0) f^k\left(\dfrac{t_1}{1 - t_0}, \ldots, \dfrac{t_{k+1}}{1 - t_0}\right) & \text{when } t_0 \neq 1, \\ w & \text{when } t_0 = 1, \end{cases} \tag{3}$$

where w is a point from B, and t_i are the barycentric coordinates of a point from σ^{k+1}.

Extending D_k by linearity to the whole group $C^s_k(B; G)$, we obtain the homomorphism

$$D_k : C^s_k(B; G) \to C^s_{k+1}(B; G).$$

It follows from equality (3) that the homomorphisms D_k and differentials ∂^s_k are related as follows:

$$\begin{aligned} \partial^s_{k+1} D_k &= 1_{C^s_k(X;G)} - D_{k-1}\partial^s_k \quad \text{when} \quad k > 0, \\ \partial^s_1 D_0 f^0 &= f^0 - h^0, \end{aligned} \tag{4}$$

where the singular simplex h^0 maps σ^0 into the point w from B.

Let $z_k \in \operatorname{Ker} \partial^s_k$, $k > 0$. Then due to (4), we have $\partial^s_{k+1} D_k z_k = z_k$, whence $z_k \in \operatorname{Im} \partial^s_{k+1}$. Thus, $H^s_k(B; G) = 0$ when $k > 0$. Similarly, the 0-dimensional cycle f^0 is homologous to the cycle h^0, therefore, $H^s_0(B; G) = G$. ■

The method used in the proof of Lemma 1 is quite useful. We now give the following definition.

Let C_*, C'_* be chain complexes, $\varphi_*, \psi_* : C_* \to C'_*$ homomorphisms. A system of homomorphisms $\{D_k\}$

$$D_k : C_k \to C'_{k+1}$$

such that the relation

$$\partial'_{k+1}D_k + D_{k-1}\partial_k = \psi_k - \varphi_k, \; D_{-1} \overset{\text{def}}{=} 0 \tag{5}$$

holds is called a *chain homotopy* connecting φ_* and ψ_*.

The homomorphisms of this relation are shown in the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{k+1} \arrow[r, "\partial_{k+1}"] & C_k \arrow[r, "\partial_k"] \arrow[dl, "D_k"'] \arrow[d, "\psi_k-\varphi_k"] & C_{k-1} \arrow[r] \arrow[dl, "D_{k-1}"] & \cdots \\
\cdots \arrow[r] & C'_{k+1} \arrow[r, "\partial'_{k+1}"] & C'_k \arrow[r, "\partial'_k"] & C'_{k-1} \arrow[r] & \cdots
\end{tikzcd}$$

The homomorphisms φ_* and ψ_* are said to be *chain-homotopic*. If $\{D_k\}$ is a chain homotopy connecting φ_* and ψ_*, then for $z_k \in \operatorname{Ker} \partial_k$, we have

$$(\psi_k - \varphi_k)z_k = \partial'_{k+1}D_k z_k \in \operatorname{Im} \partial'_{k+1}.$$

Hence, the homology group homomorphisms induced by the chain homomorphisms φ_* and ψ_* coincide.

Exercise 5°. Let the chain homomorphisms $\varphi_*, \psi_* : C_* \to C'_*$ and systems of homomorphisms $\{D_k^1\}, \{D_k^2\}, D_k^i : C_k \to C'_{k+1}, i = 1, 2$ be such that $\partial'_{k+1}D_k^1 + D_{k-1}^2\partial_k = \psi_k - \varphi_k$. Show that the homology group homomorphisms induced by the homomorphisms φ_* and ψ_* coincide.

We show that homotopic mappings of topological spaces induce chain-homotopic homomorphisms of singular chain complexes. Let us apply the following construction. Let X be a topological space, $X \times I$ a cylinder over it; it is natural to call mappings $\alpha^X, \beta^X : X \to X \times I$ defined by the formulae

$$\alpha^X(x) = (x, 0), \quad \beta^X(x) = (x, 1)$$

the lower and upper bases of the cylinder. It is evident that α^X and β^X are homotopic.

LEMMA 2. *For any space X, there exists a chain homotopy $\{D_k^X\}$ connecting α_*^X and β_*^X, i.e.,*

$$\beta_k^X - \alpha_k^X = D_{k-1}^X\partial_k^s + \partial_{k+1}^s D_k^X. \tag{6}$$

PROOF. We construct a chain homotopy $\{D_k^X : C_k^s(X; G) \to C_{k+1}^s(X \times I; G)\}$ by induction on k.

For $k = 0$, we put $D_0^X f^0 = f^0 \times 1_I$, where the singular simplex $f^0 \times 1_I$ is defined by the formula

$$f^0 \times 1_I(t_0, t_1) = (f^0(1), t_1),$$

and extend $D_0^X f^0$ to $C_0^s(X; G)$ by linearity.

Assume, for $k > 0$, that the homomorphisms D_m^X have already been defined when $m < k$ for any X, and that they are functorial.

Consider the chain

$$c_k \in C_k^s(\sigma^k \times I; G), \quad c_k = \beta_k^{\sigma^k}(1_{\sigma^k}) - \alpha_k^{\sigma^k}(1_{\sigma^k}) - D_{k-1}^{\sigma^k}\partial_k^s(1_{\sigma^k}),$$

where 1_{σ^k} is considered as a singular simplex. By the induction hypothesis,

$$\partial_k^s c_k = (\beta_{k-1}^{\sigma^k} - \alpha_{k-1}^{\sigma^k} - \partial_k^s D_{k-1}^{\sigma^k})\partial_k^s(1_{\sigma^k}) = D_{k-2}^{\sigma^k}\partial_{k-1}^s\partial_k^s(1_{\sigma^k}) = 0;$$

therefore, $c_k \in \operatorname{Ker}\partial_k^s \subset C_k^s(\sigma^k \times I; G)$. But $\sigma^k \times I$ is a convex subset of the Euclidean space; by Lemma 1, $H_k^s(\sigma^k \times I; G) = 0$. Therefore $c_k \in \operatorname{Im}\partial_{k+1}^s$, i.e., there exists a chain $u_{k+1} \in C_{k+1}^s(\sigma^k \times I; G)$ such that $\partial_{k+1}^s u_{k+1} = c_k$.

Put $D_k^{\sigma^k}(1_{\sigma^k}) = u_{k+1}$.

Now, let $f^k : \sigma^k \to X$ be a singular simplex of the space X. We define the chain $D_k^X f^k$ by the equality

$$D_k^X f^k = (f^k \times 1_I)_{k+1} D_k^{\sigma^k} 1_{\sigma^k} = (f^k \times 1_I)_{k+1} u_{k+1},$$

where $(f^k \times 1_I)(x, t) = (f^k(x), t)$, $x \in \sigma^k$, $t \in I$. Since f_k and ∂_k are commuting and D_{k-1}^X functorial, we obtain

$$\begin{aligned}\partial_{k+1}^s D_k^X f^k &= (f^k \times 1_I)_k(\beta_k^{\sigma^k} - \varphi_k^{\sigma^k} - D_{k-1}^{\sigma^k}\partial_k^s)(1_{\sigma^k}) \\ &= \beta_k^X f^k - \alpha_k^X f^k - D_{k-1}^X \partial_k^s f^k.\end{aligned}$$

Extending D_k^X by linearity to $C_k^s(X; G)$, we obtain the required homomorphism D_k^X. ■

We stress the point that the construction of $\{D_k^X\}$ is functorial, i.e., for any continuous mapping $\varphi : X \to Y$, the following diagram is commutative

$$\begin{tikzcd} C_k^s(X;G) \arrow[r, "D_k^X"] \arrow[d, "\varphi_k"] & C_{k+1}^s(X \times I;G) \arrow[d, "(\varphi \times 1_I)_{k+1}"] \\ C_k^s(Y;G) \arrow[r, "D_k^Y"] & C_{k+1}^s(Y \times I;G) \end{tikzcd}$$

THE PROOF OF THEOREM 1. Let $F : X \times I \to Y$ be a homotopy connecting φ and ψ. We define the chain homotopy

$$\{D_k : C_k^s(X; G) \to C_{k+1}^s(Y; G)\}$$

connecting φ_* and ψ_* as the family of superpositions $\{D_k = F_{k+1} D_k^X\}$ of homomorphisms of the sequence

$$C_k^s(X; G) \xrightarrow{D_k^X} C_{k+1}^s(X \times I; G) \xrightarrow{F_{k+1}} C_{k+1}^s(Y; G).$$

The statement of the theorem follows from the fact that chain-homotopic homomorphisms of chain complexes induce the same homomorphisms of homology groups. ■

COROLLARY. *A homotopy equivalence induces a homology group isomorphism.*

Thus, homotopy equivalent spaces (in particular, homeomorphic) possess the same (isomorphic) homology groups.

Exercises.

6°. Show that if X is a contractible space, then $H_0^s(X; G) = G$, $H_k^s(X; G) = 0$ for $k > 0$.

7°. Show that for the homology groups of a disjoint union $X \cup Y$, the isomorphism is held

$$H_k^s(X \cup Y; G) = H_k^s(X; G) \oplus H_k^s(Y; G).$$

Show that $H_0^s(S^0; G) = G \oplus G$, $H_k^s(S^0; G) = 0$ when $k > 0$.

8°. Show that if X and Y are path-connected (see Sec. 10, Ch. II), then $H_0^s(X; G) = G = H_0^s(Y; G)$, and any continuous mapping $\varphi : X \to Y$ induces the isomorphism

$$\varphi_{*0} : H_0^s(X; G) = H_0^s(Y; G).$$

Let X_0 be a subspace of X, $i : X_0 \to X$ an embedding mapping. Putting

$$C_k^s(X, X_0; G) = C_k^s(X; G)/C_k^s(X_0; G),$$

we have, due to Sec. 2, an exact sequence of chain complexes

$$0 \to C_*^s(X_0; G) \xrightarrow{i_*} C_*^s(X; G) \xrightarrow{j_*} C_*^s(X, X_0, G) \to 0.$$

The homology groups of the complex $C_*^s(X, X_0; G)$ are called the *singular homology groups of the pair* (X, X_0) and denoted by

$$H_*^s(X, X_0; G) = \{H_k^s(X, X_0; G)\}_{k \geq 0}.$$

It follows immediately from Lemma of Sec. 2 that the homology sequence

$$\ldots \to H_{k+1}^s(X, X_0; G) \xrightarrow{\partial_{k+1}} H_k^s(X_0; G) \xrightarrow{i_{*k}} H_k^s(X; G)$$
$$\xrightarrow{j_{*k}} H_k^s(X, X_0; G) \xrightarrow{\partial_k} \ldots \to H_0^s(X, X_0) \to 0 \quad (7)$$

is exact. Exact homology sequences are the main tool in homology theory.

Let us introduce the following more vivid description of the homology groups of a pair which follows from the above definitions. Let in the chain $\gamma = \sum_i g_i f_i^k$, all similar terms be collected, i.e., let all singular simplexes f_i^k be pairwise different, and all the coefficients g_i other than zero; we call the subset of the space X, equal to the union of images of all mappings f_i^k involved in γ with nonzero coefficients, the *support of the chain* γ. According to the definition of a quotient complex (see Sec. 2), the element $\hat{z}_k$ of the kernel Ker $\hat{\partial}_k$ of the boundary homomorphism $\hat{\partial}_k : C_k^s(X, X_0; G) \to C_{k-1}^s(X, X_0; G)$ is a coset consisting of all chains $C_k^s(X, X_0; G)$ such that (i) two different representatives of the element $\hat{z}_k$ differ only by an addend from $i_k C_k^s(X_0; G) \subset C_k^s(X; G)$, i.e., by a chain with support in the subspace X_0 ($\hat{z}_k$ being a coset); (ii) the support of the boundary $\partial_k z_k$ of any representative z_k of the element $\hat{z}_k$ is contained in X_0 ($\hat{z}_k$ is a cycle 'modulo X_0'). Similarly, the element $\hat{b}_k$ of the image of the homomorphism $\hat{\partial}_{k+1}^s$ is a coset consisting of all chains b_k in $C_k^s(X; G)$ such that (i) two representatives differ by a chain with support in X_0, (ii) each representative b_k of the coset $\hat{b}_k$ can be written in the form $\partial_{k+1}^s \gamma_{k+1}$

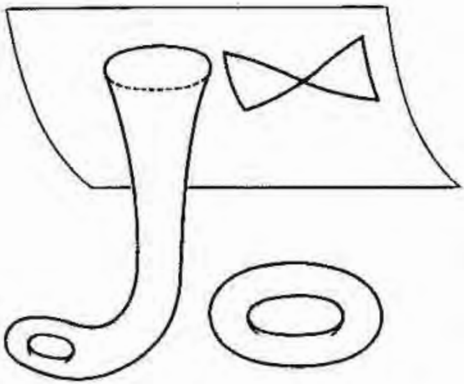


Fig. 112

$+ \gamma_k^0$, where γ_{k+1} is a certain chain from $C_k^s(X; G)$, and γ_k^0 is any chain with support in X_0. Thus, elements of homology groups are relative ('modulo the subspace X_0') cycles $\hat{z}_k$ considered up to the relative ('modulo X_0') boundaries $\hat{b}_k$. An example of a two-dimensional relative cycle is given in Fig. 112.

The connecting homomorphism δ_k associates the element $\hat{h}_k$ from the homology group of the pair $H_k(X, X_0; G)$ with the element h_{k-1}^0 from the homology group $H_{k-1}(X_0; G)$ of the subspace X_0 according to the following rule derived from the definition of a connecting homomorphism for chain complexes (see the end of Sec. 2). Let a relative cycle $\hat{z}_k \in C_k^s(X, X_0; G)$ be the representative of the element $\hat{h}_k$, and the cycle $z_k \in C_k^s(X; G)$ the representative of the element $\hat{z}_k$ treated as a coset. Consider the chain $\partial_k z_k$ from $C_{k-1}^s(X; G)$. It is clear that (i) the support of the chain $\partial_k z_k$ is contained in X_0 and therefore the chain $\partial_k z_k$ can be regarded as a chain from $C_{k-1}(X_0; G)$; (ii) since $\partial_{k-1}\partial_k z_k = 0$, the chain $\partial_k z_k$ is a cycle in $C_*(X_0; G)$. Generally speaking, the cycle $\partial_k z_k$ is not a boundary in $C_*(X_0; G)$, because z_k may not belong to $i_k C_k(X_0; G) = C_k(X_0; G)$, i.e., the support of the chain z_k may not lie in X_0. The homology class h_{k-1}^0 of the cycle $\partial_k z_k$ in $H_{k-1}(X_0; G)$ is, precisely, the image $\partial_k \hat{h}_k$ of the homology class $\hat{h}_k$. It is obvious that an arbitrary choice of the representatives $\hat{z}_k$ and z_k merely leads to the difference $\partial_k u_k - \partial_k v_k$ being a boundary in $C_*^s(X_0; G)$, and not only in $C_*(X; G)$, for two different chains u_k and v_k determining the same class $\hat{h}_k$. Therefore the definition of the homology class h_{k-1}^0 is valid.

Exercises.

9°. Let $*$ be a point from X. Show that $H_k^s(X; G) = H_k^s(X, *; G)$ for $k > 1$.

10°. Let the embedding $i: X_0 \to X$ be a homotopy equivalence. Show that $H_k^s(X, X_0; G) = 0$ for each k.

Note that generally speaking the assertion that the Mayer-Vietoris sequence is exact (see Sec. 3) is incorrect for singular homology groups. (Why? Try to give a counterexample.) However, if K_1 and K_2 are subcomplexes of a simplicial complex K then for singular homology groups of the spaces $|K_1|$ and $|K_2|$, the Mayer-Vietoris sequence is exact.

We now dwell on the barycentric subdivision of singular simplexes. Consider the barycentric subdivision of a standard simplex σ^k. Denote by $<c^{i_0}, c^{i_1}, \ldots, c^{i_q}>$ the composition of a linear (in barycentric coordinates) mapping of a standard simplex σ^q onto the simplex $(c^{i_0}, c^{i_1}, \ldots, c^{i_q})$ from the barycentric subdivision of σ^k, which sends the j-th vertex of the standard simplex to the j-th vertex c^{i_j} from the set $[c^{i_0}, c^{i_1}, \ldots, c^{i_q}]$, and the embedding mapping of the simplex $(c^{i_0}, c^{i_1}, \ldots, c^{i_q})$ into the simplex σ^k.

Note that the identity mapping 1_{σ^k} of the simplex σ^k can be regarded as an element of the group $C_k^s(\sigma^k, G)$ whose boundary is of the form $\partial_k 1_{\sigma^k} = \sum_{i=0}^{k} (-1)^i \Delta_i^{k-1}$ (see item 1).

Now, let X be an arbitrary topological space.

We define the homomorphisms

$$\Omega_k : C_k^s(X; G) \to C_k^s(X; G), \quad k = 0, 1, \ldots,$$

inductively by having put

$$\Omega_0 = 1_{C_0^s(X;G)}. \tag{8a}$$

Assume that the homomorphisms Ω_{k-1} have already been defined for an arbitrary topological space X and, moreover, that for the singular simplex 1_{σ^k} of the space σ^k, the chain $\Omega_{k-1}(\partial_k^s 1_{\sigma^k})$ can be represented in the form

$$\Omega_{k-1}(\partial_k^s 1_{\sigma^k}) = \sum_j g_j \cdot <c_j^0, c_j^1, \ldots, c_j^{k-1}>, \tag{8b}$$

where c_j^r is a vertex of the barycentric subdivision of the $(k-1)$-dimensional faces of the simplex σ^k. It is obvious that this requirement is fulfilled when $k - 1 = 0$. Now, put

$$\Omega_k(1_{\sigma^k}) = \sum_j g_j \cdot <b^{0,1,\ldots,k}, c_j^0, c_j^1, \ldots, c_j^{k-1}>, \tag{8c}$$

where $b^{0,1,\ldots,k}$ is the barycentre of the simplex σ^k, and g_j and c_j^r are the same as in (8b).

Let us define the homomorphism Ω_k on the singular simplex $f^k : \sigma^k \to X$ for an arbitrary space X.

Let $f_*^k : C_*^s(\sigma^k; G) \to C_*^s(X; G)$ be a chain complex homomorphism induced by the mapping $f^k : \sigma^k \to X$. Put

$$\Omega_k(f^k) = f_k^k \Omega_k(1_{\sigma^k}). \tag{8d}$$

Extending Ω_k to $C_k^s(X; G)$ by linearity

$$\Omega_k\Big(\sum_i g_i f_i^k\Big) = \sum_i g_i \Omega_k(f_i^k), \tag{8e}$$

we complete the definition of Ω_k. It is clear that the chain $\Omega_k(\partial^s_{k+1} 1_{\sigma^{k+1}})$ admits a representation similar to (8b). The inductive construction of Ω_k is thus complete.

Thus, the chain $\Omega_k(f^k)$ is given rise as the sum of the restrictions of the mapping f^k to the k-dimensional simplexes of the barycentric subdivision of the simplex σ^k.

The homomorphisms Ω_k are functorial and commute with the differentials $\partial_k \Omega_k = \Omega_k \partial_k$ (verify!). The collection of the homomorphisms Ω_k forms the homomorphism $\Omega_* : C^s_*(X; G) \to C^s_*(X; G)$ of the complex $C^s_*(X; G)$ into itself.

Exercises.

11°. Show that the homomorphisms Ω_* and $1_{C^s_*(X;G)}$ are chain-homotopic.

12°. Let A and B be closed, disjoint subspaces of a normal Hausdorff space X. Show that for any cycle $z_k \in C^s_k(X; G)$, there exists a cycle $(\Omega_k)^r z_k$ homologous to it such that for any singular simplex f^k of the cycle $(\Omega_k)^r z_k$, its image does not intersect both A and B simultaneously.

3. Homology and Homotopy Groups.

It is natural to attempt to establish a relation between the singular homology groups and homotopy groups of a space. This problem turns out to be quite complicated; only partial results have been obtained. Thus, a 1-dimensional homology group of a path-connected space is completely determined by its fundamental group.

THEOREM 2°. *Let X be a path-connected space with* a *base point x_0. Then*

$$H^s_1(X, Z) = \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)], \tag{9}$$

*where $[\pi_1(X, x_0), \pi_1(X, x_0)]$ is the commutant * of the group $\pi_1(X, x_0)$.*

We outline the proof of formula (9) while dwelling on geometric ideas only. First of all, note that any loop of the space X (originating at the point x_0) is a singular cycle (the singular simplex $[0, 1] \to [0, 1]/0 \sim 1 = S^1 \xrightarrow{\alpha} X$ is a cycle). Hence, the homomorphism of the group $\pi_1(X, x_0)$ into $H^s_1(X; Z)$ which we will denote by θ is given rise.

Second, θ can be shown to be an epimorphism. In fact, each cycle in $H^s_1(X; Z)$ determines (not uniquely) several loops in the space X, possibly, starting at different points. These various loops can be transformed into one loop by joining their origins to the point x_0 by means of a path that can be circumnavigated in the forward and reverse directions (Fig. 113). The complex loop obtained at the point x_0 is transformed by the homomorphism θ into the original singular cycle. (More precisely, the class of this loop is transformed into the class of the original cycle.)

Third, the commutant of the group $\pi_1(X, x_0)$ lies in the kernel of θ. In fact, the loop $\alpha \cdot \beta \cdot \alpha^{-1} \cdot \beta^{-1}$ under the action of the homomorphism θ is transformed, roughly speaking, into the cycle $\alpha + \beta + \alpha^{-1} + \beta^{-1}$; the singular cycle group is commutative, and the cycles $\alpha + \alpha^{-1}$ and $\beta + \beta^{-1}$ are homologous to zero. Therefore, the loop $\alpha \cdot \beta \cdot \alpha^{-1} \cdot \beta^{-1}$ is transformed into a cycle which is homologous to zero.

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* Remember that the *commutant* $[\pi, \pi]$ of a group π is a subgroup generated by commutators of the form $g_1 \cdot g_2 \cdot g_1^{-1} \cdot g_2^{-1}$, where $g_1, g_2 \in \pi$. The commutant of a group is its normal subgroup.

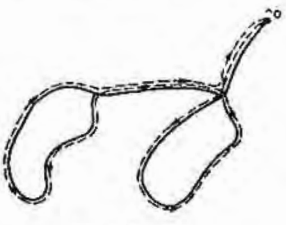

Fig. 113

It can be shown that the commutant makes up the whole kernel of the homomorphism θ (actually, the 'inverse' homomorphism of the group H_1^s into $\pi_1/[\pi_1, \pi_1]$ is constructed while proving the surjectivity).

Exercise 13°. Restore the proof of Theorem 2 by the plan given.

We give the following statement without proof.

THEOREM 3 (HUREWICZ). *Let X be a path-connected topological space such that $\pi_k(X) = 0$ when $k < q$ and $\pi_q(X) \neq 0$ $(q > 1)$. Then $H_k^s(X; Z) = 0$ when $0 < k < q$ and $H_q^s(X; Z) = \pi_q(X)$, the following diagram being commutative*

$$\begin{tikzcd} \pi_q(X) \arrow[r, "f_q"] \arrow[d, phantom, "\wr\wr"] & \pi_q(X) \arrow[d, phantom, "\wr\wr"] \\ H_q^s(X;Z) \arrow[r, "f_{*q}"'] & H_q^s(X;Z) \end{tikzcd}$$

for any mapping $f: X \to X$.

5. HOMOLOGY THEORY AXIOMS

In the two previous sections, we considered two homology theories, viz. simplicial and singular. Besides, there exist some more homology theories in algebraic topology. Historically, simplicial homology theory was introduced earlier. Different approaches as regards the construction of homology theory for general topological spaces (Alexandrov-Čech homology theory, singular homology theory, etc.) were developed later. The problem concerning the conditions for the equivalence of two different theories appeared to be quite complicated.

Quite useful in this connection is the axiomatic approach to homology theory, which implies that the basic properties of the correspondence between topological and algebraic notions are given axiomatically, and all the remaining concepts are deduced from the axioms chosen. Such a system of axioms was developed by Steenrod and Eilenberg, and here we formulate their axioms.

Homology theory H_* with a connecting homomorphism δ_* is the collection of covariant functors $\{H_k\}$, $k = 0, 1, 2, \dots$, from the category of pairs of topological

spaces (X, A), $A \subset X$, into the category of Abelian groups, and the collection of functorial homomorphisms $\{\delta_k\}$, $k = 1, 2, \ldots$,

$$\delta_k(X, A): H_k(X, A) \to H_{k-1}(A, \varnothing).$$

Moreover, the following axioms should be fulfilled:

(1) HOMOTOPY AXIOM. *Let mappings $f, g: X \to Y$ be homotopic, and $F: X \times I \to Y$ a homotopy connecting them. Let $A \subset X$ and $B \subset Y$, and $F(A \times I) \subset B$. Then*

$$H_*(f) = H_*(g): H_*(X, A) \to H_*(Y, B)$$

for arbitrary X, Y, A, B, f, g.

(2) EXACTNESS AXIOM *For any pair (X, A) and embeddings $i: (A, \varnothing) \to (X, \varnothing)$, $j: (X, \varnothing) \to (X, A)$, there is an exact sequence*

$$\ldots \xrightarrow{\delta_{k+1}(X,A)} H_k(A, \varnothing) \xrightarrow{H_k(i)} H_k(X, \varnothing) \xrightarrow{H_k(j)} H_k(X, A)$$
$$\xrightarrow{\delta_k(X,A)} H_{k-1}(A, \varnothing) \to \ldots \to H_0(X, A) \to 0. \quad (1)$$

(3) EXCISION AXIOM. *Let (X, A) be an arbitrary pair, U open in X and $\overline{U} \subset \operatorname{Int} A$. Then the embedding of pairs $j: (X \setminus U, A \setminus U) \to (X, A)$ induces the isomorphism*

$$H_*(j): H_*(X \setminus U, A \setminus U) \approx H_*(X, A). \quad (2)$$

(4) DIMENSION AXIOM. *For a space $*$ consisting of one point, $H_k(*, \varnothing) = 0$ when $k > 0$.*

Exercise 1°. Verify the fulfilment of the axioms of homology theory for singular homology theory.

The axioms of homology theory are complete in the following sense.

THE UNIQUENESS THEOREM. *Let H_* and $\overline{H}_*$ be two homology theories. If there exists an isomorphism $\varkappa_0: H_0(*, \varnothing) \approx \overline{H}_0(*, \varnothing)$, then these theories are naturally isomorphic on the category of pairs of compact polyhedra, i.e.,*

(i) *for any pair of compact polyhedra (X, A) such that a triangulation of A is a subset of a triangulation of X, and for each $k \geqslant 0$, a unique family of isomorphisms $\varkappa_k(X, A): H_k(X, A) \approx \overline{H}_k(X, A)$, $k > 0$ is defined with $\varkappa_0(*, \varnothing) = \varkappa_0$.*

(ii) *for any mapping $f: (X, A) \to (Y, B)$ of pairs of compact polyhedra and each $k \geqslant 0$, the relations $H_k(f) = \overline{H}_k(f)$ implying the commutativity of the diagrams*

$$\begin{array}{ccc} H_k(X,A) & \xrightarrow{H_k(f)} & H_k(Y,B) \\ \varkappa_k \wr\!\wr & & \wr\!\wr \varkappa_k \\ \overline{H}_k(X,A) & \xrightarrow{\overline{H}_k(f)} & \overline{H}_k(Y,B) \end{array}$$

are valid;

(iii) *the diagrams*

$$\begin{array}{ccc} H_{k+1}(X,A) & \xrightarrow{\delta_{k+1}(X,A)} & H_k(A,\varnothing) \\ \varkappa_{k+1} \wr\wr & & \wr\wr\ \varkappa_k(A,\varnothing) \\ \bar{H}_{k+1}(X,A) & \xrightarrow{\bar{\delta}_{k+1}(X,A)} & \bar{H}_k(A,\varnothing) \end{array}$$

arising under the isomorphism of exact sequences of form (1) are commutative.

Since it is beyond the elementary course, we do not give the proof of the uniqueness theorem here.

In particular, singular and simplicial theories coincide on the category of pairs of compact polyhedra. Thus, for a compact polyhedron $|K|$, the isomorphism is valid:

$$H_*(|K|;G) = H_*^s(|K|;G). \tag{3}$$

We will use this fact (the proof is not given here) in Secs. 6 and 8 while transferring from one homology theory to another.

Note that the independence of simplicial homology of a compact polyhedron from the choice of a triangulation can be established both within the scope of simplicial theory itself and with the use of singular homology theory. The latter method consists in constructing an isomorphism between the homology groups of an arbitrary triangulation of a polyhedron and the singular homology groups of this polyhedron. The particulars of this reasoning are complicated enough, and we do not give them here.

Exercise 2°. By means of the uniqueness theorem, establish the validity of the exact Mayer-Vietoris sequence for singular homology theory

$$\begin{aligned} \ldots \to H_k^s(|K_1| \cap |K_2|;G) &\to H_k^s(|K_1|;G) \oplus H_k^s(|K_2|;G) \\ &\to H_k^s(|K_1| \cup |K_2|;G) \to H_{k-1}^s(|K_1| \cap |K_2|;G) \\ &\qquad \ldots \to H_0^s(|K_1| \cup |K_2|;G) \to 0, \end{aligned} \tag{4}$$

where K_1, K_2 are subcomplexes of a finite simplicial complex K.

Note in conclusion that there exist homology theories satisfying axioms (1)-(3) but not satisfying the dimension axiom. Such homology theories are called *extraordinary*, and it is their investigation that makes up, largely, the basic topics of modern algebraic topology.

Along with homology groups, the so-called cohomology groups are used in algebraic topology. The main difference of cohomology theory from homology theory consists in the fact that cohomology theory is the collection H^* of contravariant functors H^k and therefore most arrows in cohomology theory change their directions as compared with homology theory.

The fundamental object in cohomology theory is a cochain complex C^*, i.e., the sequence

$$0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \ldots \to C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \xrightarrow{d^{k+1}} C^{k+2} \to \ldots$$

of Abelian groups C^k (cochain groups) and their homomorphisms d^k (differentials or coboundary homomorphisms) such that $d^{k+1}d^k = 0$. The cochain complex cohomology groups are the factor groups

$$H^k(C^*) = \operatorname{Ker} d^k / \operatorname{Im} d^{k-1}.$$

Cochain complexes are often obtained from chain complexes by the following method. Let C_* be a chain complex, G an Abelian group. Let $C^k = \operatorname{Hom}(C_k, G)$ be the set of all homomorphisms of the group C_k into the group G. For $\psi^k \in \operatorname{Hom}(C_k, G)$, we define the element $d^k\psi^k \in \operatorname{Hom}(C_{k+1}, G)$ by the equality

$$(d^k\psi^k)\gamma_{k+1} = \gamma^k(\partial_{k+1}\gamma_{k+1})$$

on an arbitrary element $\gamma_{k+1} \in C_{k+1}$. Thus, by the boundary homomorphisms ∂_k of the chain complex C_*, we define the coboundary homomorphisms d^k of the cochain complex C^*. It is obvious that

$$\begin{aligned}(d^{k+1}d^k\psi^k)\gamma_{k+2} &= (d^k\psi^k)(\partial_{k+2}\gamma_{k+2}) \\ &= \psi^k(\partial_{k+1}\partial_{k+2}\gamma_{k+2}) = \psi^k(0) = 0,\end{aligned}$$

so that C^* is a cochain complex indeed.

Applying this method to $C_* = C_*(K; Z)$, i.e., the chain complex of the simplicial complex K with integral coefficients, we obtain a cochain complex $C^*(K; G)$, where $C^k(K; G) = \operatorname{Hom}(C_k(K; Z), G)$. The cohomology groups $H^k(C^*(K; G))$ are called the simplicial cohomology groups of the simplicial complex K (or the polyhedron $|K|$) with coefficients in G. Similarly, by considering the singular chain complex $C_*^s(X; Z)$ as C_*, we obtain the singular cohomology groups of the topological space X with coefficients in G. However, here certain difficulties should be overcome, viz., those related to the fact that the set of singular chain group generators is infinite.

A system of axioms exists and the uniqueness theorem is valid also for cohomology theory. These are similar to the axioms and theorem for homology theory. An important advantage of cohomology theories over homology theories is that the cohomology groups of geometric objects produce a ring with a multiplication which is, generally speaking, nontrivial. In a number of problems, both homology and cohomology groups are used.

6. HOMOLOGY GROUPS OF SPHERES. DEGREE OF MAPPING

1. The Homology Groups of Spheres. We now calculate the singular homology groups of the spheres S^n. The knowledge of these groups enables us to introduce the notions of the degree of a mapping, characteristic and the index of a singular point of a vector field.

Let X be a cell complex, Y a finite subcomplex. We show that

$$H_k^s(X, Y; G) = H_k^s(X/Y; G) \tag{1}$$

when $k > 0$, and where X/Y is a factor space of X relative to Y.

Note, first, that the cell complex X/Y is homotopy equivalent to the complex $X \cup_i CY$, where CY is the cone * over Y with the vertex $*$, and $i: Y \to X$ is the embedding. In fact, the complex X/Y coincides with the complex $(X \cup_i CY)/CY$. Since CY is a contractible subcomplex of the complex $X \cup_i CY$, the complexes $(X \cup_i CY)/CY$ and $X \cup_i CY$ are homotopy equivalent (see Ex. 7, Sec. 10, Ch. IV). Therefore

$$H_k^s(X/Y; G) = H_k^s(X \cup_i CY; G),$$

and when $k > 0$,

$$H_k^s(X/Y; G) = H_k^s(X \cup_i CY, *; G)$$

(see Ex. 9, Sec. 4).

The cone CY is homotopy equivalent to the point $* \in CY$, hence

$$H_k^s(X \cup_i CY, *; G) = H_k^s(X \cup_i CY; G).$$

Consider the embedding mapping of pairs

$$I: (X, Y) \to (X \cup_i CY, CY);$$

it induces the homomorphism

$$I_*: H_*^s(X, Y; G) \to H_*^s(X \cup_i CY, CY; G).$$

Let us show that I_* is an isomorphism. We break the cone CY into two parts C^1Y and C^2Y, as shown in Fig. 114. It is obvious that

$$H_*^s(X \cup_i C^2Y, C^2Y; G) = H_*^s(X, Y; G).$$

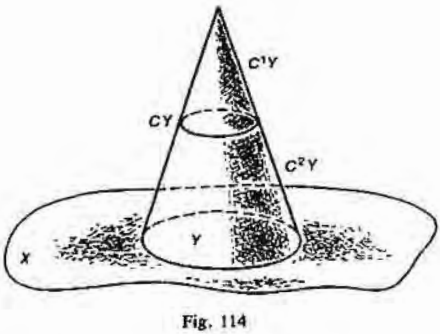

Fig. 114

* Remember that for a topological space Y, the cone CY is defined as the factor space $(Y \times I)/(Y \times 0)$.

Each cycle $z_k \in C_k^s(X \cup_f CY, CY; G)$ can be replaced by a cycle $(\Omega_k)^r z_k$ homologous to it and such that the image of each singular simplex from $(\Omega_k)^r z_k$ intersecting X will not intersect $C^1 Y$, and, conversely, each singular simplex intersecting $C^1 Y$ will not intersect X (see Ex. 12, Sec. 4). Discarding in the chain $\Omega^{(r)} z_k$ all simplexes intersecting $C^1 Y$, we will obtain a cycle $z_k' \in C_k(X \cup_f CY, CY; G)$ which is homologous to the original. On the other hand, z_k' can be considered as a cycle in the group of chains $C_k^s(X \cup_f C^2 Y, C^2 Y; G)$; therefore, I_k is an epimorphism.

It can be shown similarly that I_k is a monomorphism.

Consider an application of formula (1) to the calculation of homology groups of the sphere S^n. We shall need the homology groups of the disc $\bar{D}^n$. Since $\bar{D}^n$ is contractible to a point, the homology groups of the disc are isomorphic to the homology groups of the point, viz.,

$$H_k^s(\bar{D}^n; G) = \begin{cases} G & \text{when} \quad k = 0 \\ 0 & \text{when} \quad k > 0 \end{cases}$$

(see Ex. 6, Sec. 4). We begin calculating with small dimensions n. Since S^0 is the disjoint union of two points,

$$H_0^s(S^0; G) = G \oplus G, \quad H_k^s(S^0; G) = 0 \quad \text{when} \quad k > 0.$$

Furthermore, due to the path-connectedness of S^n, when $n > 0$, we have $H_0^s(S^n; G) = G, n > 0$.

Note now that the sphere S^n is homeomorphic to the factor space $\bar{D}^n/S^{n-1}$. Therefore, due to (1), we have

$$H_k^s(\bar{D}^n, S^{n-1}; G) = H_k^s(S^n; G) \quad \text{when} \quad k > 0.$$

Let us make use of this result.

Consider an exact homology sequence of $\bar{D}^1, S^0$, while replacing, when $k > 0$, the homology groups of the pair by the homology groups of the circumference S^1:

$$\begin{aligned} \ldots \to H_k^s(S^0; G) \to H_k^s(\bar{D}^1; G) \to H_k^s(S^1; G) \to H_{k-1}^s(S^0; G) \\ \to \ldots \to H_1^s(S^0; G) \to H_1^s(\bar{D}^1; G) \to H_1^s(S^1; G) \to H_0^s(S^0; G) \\ \to H_0^s(\bar{D}^1; G) \to H_0^s(\bar{D}^1, S^0; G) \to 0. \quad (2) \end{aligned}$$

Having noticed that $H_k^s(\bar{D}^1; G) = 0$ when $k \geqslant 1$ and $H_{k-1}^s(S^0; G) = 0$ when $k > 1$, we obtain from (2) a short exact sequence

$$0 \to H_k^s(S^1; G) \to 0, \quad k > 1,$$

hence $H_k^s(S^1; G) = 0$ when $k > 1$. Besides, the homomorphism $H_0^s(S^0; G) \to H_0^s(\bar{D}^1; G)$ is epimorphic (verify that by definition!). Therefore, our exact sequence (2) leads to a short exact sequence

$$0 \to H_1^s(S^1; G) \to G \oplus G \xrightarrow{pr_1 + pr_2} G \to 0,$$

and hence we obtain the isomorphism $H_1^s(S^1; G) = G$.

Now we apply induction. Assume that when $1 \leqslant q \leqslant n-1$ the isomorphisms have been established for the spheres S^q,

$$H_k^s(S^q; G) \approx \begin{cases} G & \text{when } k = 0, q, \\ 0 & \text{when } k \neq 0, q. \end{cases}$$

Consider the exact homology sequence of the pair $(\bar{D}^n; S^{n-1})$ while replacing, as earlier, the homology groups of the pair by the homology groups of the sphere S^n:

$$\ldots \to H_k^s(\bar{D}^n; G) \to H_k^s(S^n; G) \to H_{k-1}^s(S^{n-1}; G) \to H_{k-1}^s(\bar{D}^n; G) \to \ldots \quad (3)$$

When $k > 1$, we have $H_k^s(\bar{D}^n; G) = 0$, $H_{k-1}^s(\bar{D}^n; G) = 0$, therefore the portion under consideration of exact sequence (3) is of the form

$$0 \to H_k^s(S^n; G) \to H_{k-1}^s(S^{n-1}; G) \to 0,$$

hence the isomorphism follows: $H_k^s(S^n; G) \approx H_{k-1}^s(S^{n-1}; G)$, $k > 1$. Thus, when $n \geqslant 2$, we obtain

$$H_2^s(S^n; G) = 0, \ldots, H_{n-1}^s(S^n; G) = 0,$$
$$H_n^s(S^n; G) \approx H_{n-1}^s(S^{n-1}; G) = G, \ldots$$

To calculate $H_1^s(S^n; G)$, we put $k = 1$ in (3):

$$\ldots \to H_1^s(\bar{D}^n; G) \to H_1^s(S^n; G) \to H_0^s(S^{n-1}; G) \xrightarrow{i_{*0}} H_0^s(\bar{D}^n; G) \to \ldots$$

Since S^{n-1}, $\bar{D}^n$ are path-connected, we have $H_0^s(S^{n-1}; G) \overset{i_{*0}}{=} H_0^s(\bar{D}^n; G) = G$ (see Ex. 8, Sec. 4). Hence, Ker $i_{*0} = 0$, and because of the exactness of (3), we obtain a short exact sequence $0 \to H_1^s(S^n; G) \to 0$, i.e., $H_1^s(S^n; G) = 0$.

The induction hypothesis is thus extended to $q = n$. Therefore, we have ultimately:

$$\begin{aligned} &H_0^s(S^n; G) = G; \quad H_j^s(S^n; G) = 0, \quad j \neq 0, \quad n \geqslant 1; \\ &H_n^s(S^n; G) = G, \quad n \geqslant 1; \quad H_0^s(S^0; G) = G \oplus G; \\ &H_j^s(S^0; G) = 0, \quad j \geqslant 1. \end{aligned} \quad (4)$$

Thus, the homology groups of S^n have been computed.

While calculating the homology groups of S^n, we did not use the uniqueness theorem of homology theory (see Sec. 5). We could have used it as follows: since the sphere S^n is homeomorphic to the boundary $\partial\tau^{n+1}$ of the simplex τ^{n+1}, we have the isomorphism

$$H_i(|[\partial\tau^{n+1}]|; G) = H_i^s(S^n; G), \quad (5)$$

whence, due to the results (see (9), Sec. 3) concerning $H_*([\partial\tau^{n+1}]; G)$, we obtain the same result as in (4).

Note that in Sec. 4, Ch. III, the Brouwer fixed-point theorem and the theorem on the impossibility of forming a retraction of the n-disc onto the boundary sphere were based on the functorial property of homotopy groups and on the result which

has not been proved: $\pi_n(S^n) = Z$. Now, on the basis of the established isomorphism $H_n^s(S^n; Z) = Z$ and homology functor, the two above-mentioned important theorems may be considered as proved rigorously. In fact, their proof only involved the axioms of a functor to the category of Abelian groups and also the knowledge of the group of the space S^n.

Exercise 1°. Deduce from Theorem 3, Sec. 4, that $\pi_k(S^n) = 0, k < n; \pi_n(S^n) = Z$.

Let us discuss the topological invariance of the notion of the dimension of a Euclidean space. It is known from algebra that two Euclidean spaces of the same dimension are isomorphic, and hence homeomorphic. It is known also that the spaces R^m and R^n are not isomorphic when $m \neq n$. A question arises whether they are homeomorphic. The following theorem provides a negative answer to this question and states thereby that the dimension of a Euclidean space is a topological invariant.

THEOREM 1. *If $m \neq n$ then the spaces R^m and R^n are not homeomorphic.*

PROOF. Consider one-point compactifications $\bar{R}^m = R^m \cup \xi^m$ and $\bar{R}^n = R^n \cup \xi^n$ of the spaces R^m and $\bar{R}^n$ (see Sec. 14, Ch. II). The bases of neighbourhoods of the points ξ^m, ξ^n are the complements of closed balls with centres at the origin of coordinates on the spaces R^m and R^n, respectively. It is easy to see that a one-point compactification of a Euclidean space is homeomorphic to a sphere of the same dimension.

Assume that there exists a homeomorphism $\Phi : R^m \to R^n$. It can be extended to the mapping $\bar{\Phi}: \bar{R}^m \to \bar{R}^n$ by having put $\bar{\Phi}(\xi^m) = \xi^n$. It is easy to see that the mapping $\bar{\Phi}$ is also a homeomorphism. We obtain hence that the spheres S^m and S^n are also homeomorphic. Then due to the topological invariance of the homology groups $H_k^s(S^m; Z) = H_k^s(S^n; Z)$ for each k.

We know, however, that this is not so when $m \neq n$. Therefore, the assumption concerning the existence of the homeomorphism $\Phi : R^m \to R^n$ when $m \neq n$ is incorrect. ■

2. The Degree of a Mapping. We now pass over to the study of homomorphisms of homology groups induced by mappings of n-dimensional spheres. It follows from the path-connectedness of the sphere that if $\varphi : S_1^n \to S_2^n$ is a mapping from one replica of the sphere to another, then the homomorphism $\varphi_{*0} : H_0^s(S_1^n; G) \to H_0(S_2^n; G)$ is an isomorphism. The homomorphism

$$\varphi_{*n} : H_n^s(S_1^n; G) \to H_n^s(S_2^n; G)$$

is not, generally speaking, an isomorphism. If the group of integers Z is taken as the group of coefficients G and the isomorphisms $H_n^s(S_i^n; Z) = Z$, $i = 1, 2$ are fixed then the homomorphism φ_{*n} can be considered as the endomorphism $\varphi_{*n} : Z \to Z$ of the group Z. Such a homomorphism is determined uniquely by the value of φ_{*n} on the generating element $1 \in Z$, because $\varphi_{*n}(m) = m\varphi_{*n}(1)$.

DEFINITION 1°. The number $\varphi_{*n}(1)$ is called the *degree of a mapping* φ and denoted by $\deg \varphi$.

Note that deg φ can, generally speaking, assume any integral values. The sign of deg φ depends on the choice of the generating elements in the groups $H_n^s(S_1^n; Z)$, $H_n^s(S_2^n; Z)$, i.e., on the method of setting isomorphisms of these groups with the group Z. If γ is a generating element of the group $H_n^s(S^n; Z)$ then $(-\gamma)$ is also its generating element; thus, the isomorphism $H_n^s(S^n; Z) = Z$ can be established in two ways. If φ is a mapping of S^n into itself then deg φ does not depend on the choice of a generating element.

Note that it follows from Theorem 3, Sec. 4, that the definitions of the degree of a mapping $\varphi : S^n \to S^n$ (i.e., deg φ) which are given in terms of homotopy groups (see Sec. 4, Ch. III) and homology groups are identical.

It is obvious that if $\varphi, \psi : S^n \to S^n$ are homotopic mappings then deg φ = deg ψ. The converse is also true (the Hopf theorem) but the proof is not given here.
Exercises.
2°. Prove that for $\varphi, \psi : S^n \to S^n$, the formula deg $(\varphi\psi)$ = deg $\varphi \cdot$ deg ψ is valid.
3°. Show that the degree of the constant mapping of the sphere S^n into itself equals zero.
4°. Let a mapping $\Phi : R^{n+1} \to R^{n+1}$ be such that $\Phi(x) \neq 0$ when $r \leqslant \|x\| \leqslant R$, and mappings $\bar{\Phi}_\rho : S^n \to S^n$ be defined by the equalities

$$\bar{\Phi}_\rho(x) = \frac{\Phi(\rho x)}{\|\Phi(\rho x)\|}, \quad x \in S^n, \quad r \leqslant \rho \leqslant R.$$

Prove that deg $\bar{\Phi}_r$ = deg $\bar{\Phi}_R$.

Hint: Construct a homotopy connecting the mappings $\bar{\Phi}_r$ and $\bar{\Phi}_R$.

The following two exercises are easy to do on the basis of the isomorphism of singular and simplicial homology groups.
Exercises.
5°. Let $A : R^{n+1} \to R^{n+1}$ be a nonsingular linear operator. We define the mapping $\bar{A} : S^n \to S^n$ by the formula

$$\bar{A}(x) = \frac{A(x)}{\|A(x)\|}, \quad x \in S^n.$$

Prove that for the operator $A = -I : R^{n+1} \to R^{n+1}$, the equality deg $\bar{A}$ = $= (-1)^{n+1}$ holds.
6°. Prove that for an arbitrary nonsingular linear operator $A : R^{n+1} \to R^{n+1}$, the equality deg $\bar{A}$ = sign $|A|$ is valid.

Hint: Show that in the class of nonsingular linear operators, A is homotopic to an operator A' whose matrix is diagonal and whose diagonal elements equal ± 1, and construct a simplicial partition of the sphere which is invariant with respect to the transformation A'.

Consider a mapping $\Phi : U \to R^{n+1}$, where U is a region in R^{n+1}. While investigating the solutions of the equation

$$\Phi(x) = 0,$$

it is customary to call the mapping Φ the *vector field on U* (a point x is associated with the vector $\Phi(x)$), and the solutions of equation (6) *singular points of the vector field* Φ.

In practice, the mapping Φ is not always continuous. If it has isolated points of discontinuity (or points of indeterminacy of value), then these points are also called singular points. Most subsequent statements are also valid for such vector fields.

Let x^0 be an isolated singular point of a vector field Φ, i.e., $\Phi(x^0) = 0$, and let there be no other solutions of equation (6) in a neighbourhood of the point x^0. Then for a sufficiently small R, when $0 < r < R$, the degree of the mapping $\tilde{\Phi}_r : S^n \to S^n$ given by the equality

$$\tilde{\Phi}_r(x) = \frac{\Phi(rx + x^0)}{\|\Phi(rx + x^0)\|} \tag{7}$$

is defined, and does not depend on the choice of r (compare with Exercise 3).

DEFINITION 2. The degree $\deg \tilde{\Phi}_r$ of the mappings $\tilde{\Phi}_r$ (for sufficiently small r) is called the *index of the isolated singular point* x^0 of the vector field Φ; we will denote it by $\operatorname{ind}(x^0, \Phi)$.

Let a field Φ have no singular points on the boundary $S_r^n(x^0)$ of the ball $D_r^{n+1}(x^0)$ with radius r and centre at the point x^0 (it is not assumed now that x^0 is a singular point and r small). It is evident that in this case also, formula (7) defines the mapping $\tilde{\Phi}_r : S^n \to S^n$.

DEFINITION 3. The degree $\deg \tilde{\Phi}_r$ of a mapping $\tilde{\Phi}_r$ is called the *characteristic of the vector field* Φ on the boundary of the ball $D_r^{n+1}(x^0)$. We will denote the characteristic by $\chi(\Phi, S_r^n(x^0))$.

Along with the term 'characteristic of a vector field', the term 'rotation of a vector field' is often used, which is similar to the 2-dimensional case, where for $\varphi : S^1 \to S^1$, the degree $\deg \varphi$ is the algebraic number of rotations of the vector $\varphi(x)$ when x ranges over the circumference S^1 (in the positive direction).

THEOREM 2. *Let a field* Φ *have no singular points in a closed ball* $\bar{D}_r^{n+1}(x^0)$, *then* $\chi(\Phi, S_r^n(x^0)) = 0$.

PROOF. The mapping $\tilde{\Phi}_r$ is homotopic to the constant mapping Φ_0 of S^n into the point $\dfrac{\Phi x^0}{\|\Phi x^0\|} \in S^n$, the degree of Φ_0 being zero. The corresponding homotopy is given, e.g., by the formula

$$\Phi(t, x) = \frac{\Phi(trx + x^0)}{\|\Phi(trx + x^0)\|}, \quad 0 \leqslant t \leqslant 1, \quad x \in S^n. \blacksquare$$

COROLLARY. *If* $\chi(\Phi, S_r^n(x^0)) \neq 0$ *then the field* Φ *has at least one singular point in the ball* $D_r^{n+1}(x^0)$.

Note that the characteristic $\chi(\Phi, S_r^n(x^0))$ is defined even if the field Φ is given only on the boundary $S_r^n(x^0)$ of the ball $D_r^{n+1}(x^0)$ *.

The following theorem is a direct corollary to Theorem 2.

THEOREM 3. *Let a field Φ be given on the sphere $S_r^n(x^0)$ and have no singular points. If $\chi(\Phi, S_r^n(x^0)) \neq 0$ then Φ cannot be extended to the ball $\bar{D}_r^{n+1}(x^0)$ without singular points.*

The converse to Theorem 3 is also valid; it follows from the above-mentioned Hopf theorem.

The characteristic of a vector field Φ can be defined on the boundary of any region $\bar{\Omega} \subset R^{n+1}$ which is a compact polyhedron provided that $\Phi(x) \neq 0$ on $\partial\bar{\Omega}$.

The following theorem which we give without proof relates the global characteristic $\chi(\Phi, \partial\bar{\Omega})$ of a vector field Φ with the local characteristics, viz., indices ind (x^i, Φ) of the singular points of the field Φ.

THEOREM 4. *Let a field Φ have no singular points on $\partial\Omega$, and have a finite number m of singular points $x^1, \dots, x^m$ in $\bar{\Omega}$; then*

$$\chi(\Phi, \partial\bar{\Omega}) = \sum_{i=1}^{m} \operatorname{ind}(x^i, \Phi). \tag{8}$$

Exercise 7°. Let x^0 be a singular point of a smooth vector field Φ on $U \subseteq R^{n+1}$, and let the Jacobian matrix $\left(\frac{\partial\Phi}{\partial x}\right)$ of the mapping Φ at the point x^0 be nonsingular (such points are said to be nondegenerate **). Prove that x^0 is an isolated singular point of the field Φ and that

$$\operatorname{ind}(x^0, \Phi) = \operatorname{sign}\det\left(\frac{\partial\Phi}{\partial x}\right)\Big|_{x^0}.$$

Hint: Construct a homotopy connecting the mappings

$$\Phi_r: S^n \to S^n \quad \text{and} \quad \left(\frac{\partial\Phi}{\partial x}\right)\Big|_{x_0}: S^n \to S^n.$$

Note that here ind (x^0, Φ) coincides with the 'sign' of the point $x_0 \in \Phi_0^{-1}$ of the inverse image of a regular value (see Sec. 6, Ch. IV). Therefore, for a smooth field Φ, provided that the singular points $x^1, \dots, x^m$ are nondegenerate, formula (8) coincides with the definition of the oriented degree of a mapping $\Phi: \Phi^{-1}(N) \to N$, where N is a connected component of the open set $R^{n+1} \setminus \partial\Omega$ and contains the point 0. Moreover, the manifolds $\Phi^{-1}(N)$ and N are oriented similarly by choosing an orientation in R^{n+1}. The indicated construction can serve as another method of determining the characteristic $\chi(\Phi, \partial\Omega)$ in the case a field Φ is smooth on Ω when (if

* The field $\Phi: S_r^n(x^0) \to R^{n+1}$ is not, evidently, a vector field on the manifold $S_r^n(x^0)$ in the sense of Sec. 8, Ch. IV.

** These are regular points of the mapping Φ (see Sec. 1, Ch. IV).

necessary) the point 0 is replaced by a sufficiently near regular value of the mapping Φ.

Exercise 8°. Let a mapping $\Phi : C \to C$ be defined by the formula $\Phi(z) = z^n$, where $n > 0$ is an integer. Considering Φ as a mapping $\Phi : R^2 \to R^2$, calculate the index of the zero singular point of the field Φ. Do the same for the mapping $\Psi(z) = (\bar{z})^n$.

Consider a vector field $X(x)$ on a manifold M^n. Let $x^0 \in M^n$ be an isolated singular point of the field $X(x)$, i.e., $X(x^0) = 0$, and let there exist a neighbourhood $U(x^0) \subset M^n$ of the point x^0, in which $X(x) \neq 0$ when $x \neq x^0$. In local coordinates, the field $X(x)$ is of the form

$$\left(x_1, \ldots, x_n; X_1(x_1, \ldots, x_n) \frac{\partial}{\partial x_1} + \ldots + X_n(x_1, \ldots, x_n) \frac{\partial}{\partial x_n}\right).$$

The index ind (x^0, X) of a singular point x^0 of a vector field $X(x)$ on a manifold can be defined as the index of a singular point $(x_1^0, \ldots, x_n^0)$ (here, x_i^0 are the coordinates of the point x^0) of the vector field

$$\Phi = \{X_1(x_1, \ldots, x_n), \ldots, X_n(x_1, \ldots, x_n)\}$$

in the space R^n.

Exercises.

9°. Prove that the index ind (x^0, X) does not depend on the choice of local coordinates.

10°. Let f be a smooth function on a manifold, x^0 a nondegenerate critical point of index λ of the function f (see Sec. 11, Ch. IV). Prove that ind $(x^0, \operatorname{grad} f(x)) = (-1)^\lambda$.

7. HOMOLOGY GROUPS OF CELL COMPLEXES

We now pass over to the study of homology groups of spaces having the homotopy type of a cell complex. This class of spaces is interesting: first, because it is quite large (see Sec. 12, Ch. IV), and second, because the cell complex homology groups can be calculated by quite a simple and elegant method.

Let X be a finite cell complex. We construct a chain complex $\check{C}_*(X; G)$ in the following way. Let us take the Abelian group of formal linear combinations $\sum_i g_i \cdot \tau_i^k$, where $g_i \in G$ are arbitrary elements, and τ_i^k k-dimensional cells of the complex X, as the group $\check{C}_k(X; G)$; the summation is over all k-dimensional cells. Therefore, the group $\check{C}_k(X; G)$ is isomorphic to the direct sum of as many replicas of the group G as there are cells of dimension k in the cellular decomposition of X. We will assume, moreover, that each replica of G corresponds to one of the k-dimensional cells.

Let us define the differential $\check{\partial}_k : \check{C}_k(X; G) \to \check{C}_{k-1}(X; G)$. Let τ^k be a k-dimensional cell of X; its boundary is contained in the union of cells of dimensions not higher than $(k - 1)$ (*$(k - 1)$-dimensional skeleton of X* denoted by X^{k-1}). It follows from the definition of a cell complex, that the cell τ^k is given by

the gluing mapping $f: S^{k-1} \to X^{k-1}$. Consider the composition $S^{k-1} \to X^{k-1} \to X^{k-1}/X^{k-2}$, where the last arrow denotes a mapping to the wedge of spheres. The space X^{k-1}/X^{k-2} is a cell complex; it consists of one cell of dimension zero, viz., the point $*$ into which the space X^{k-2} was transformed under the factorization, and of as many cells of dimension $k-1$ glued along the boundaries to the point $*$, as there were in the skeleton X^{k-1}, i.e., in X. Such a space is called the wedge of $(k-1)$-dimensional spheres. We pick a cell τ_j^{k-1} in X^{k-1}. In the wedge of spheres X^{k-1}/X^{k-2}, there corresponds to it a certain sphere S_j^{k-1}. Consider the composition of mappings

$$S^{k-1} \xrightarrow{f} X^{k-1} \to X^{k-1}/X^{k-2} \to S_j^{k-1},$$

where the last arrow means a mapping with respect to the subspace of the space X^{k-1}/X^{k-2} consisting of all spheres except S_j^{k-1}. The degree of this composition mapping is called the *incidence coefficient of the cells* τ^k and τ_j^{k-1} and denoted by $[\tau^k, \tau_j^{k-1}]$; the incidence coefficient shows how many times the boundary of the cell τ^k is 'twisted' onto the cell τ_j^{k-1} in gluing the cell τ^k to the skeleton X^{k-1}. Denote the set of cells of dimension $k-1$ in the cell complex X by Ω^{k-1}. For any cell τ^k, we define the differential $\tilde{\partial}_k$ by the formula

$$\tilde{\partial}_k(\tau^k) = \sum_{j \in \Omega^{k-1}} [\tau^k, \tau_j^{k-1}] \cdot \tau_j^{k-1}$$

and extend $\tilde{\partial}_k$ to $\tilde{C}_k(X; G)$ by linearity *. It can be shown that $\tilde{\partial}_{k-1}\tilde{\partial}_k = 0$.

Thus, a chain complex $\tilde{C}_*(X; G)$ has been constructed. Its homology groups turn out to coincide with the singular homology groups of the complex X. The proof of this fact involves only the exact sequence technique. Since it is too long we do not give it here.

The advantage of the method of computing homology groups with the help of the complex $\tilde{C}_*(X; G)$ is obvious: the groups $\tilde{C}_k(X; Z)$ have a finite number of generators in contrast with the groups $C_k^s(X; Z)$. Therefore, the subgroups of k-dimensional cycles and boundaries also have a finite number of generators as well as their factor group $H_k^s(X; Z)$. It follows from Abelian group theory that

$$H_k^s(X; Z) = \underbrace{(Z \oplus \ldots \oplus Z)}_{\rho_k} \oplus Z_{\rho_1^k} \oplus \ldots \oplus Z_{\rho_{s_k}^k},$$

where $Z_{\rho_i^k}$ is a finite cyclic group of order ρ_i^k, with ρ_i^k being divisible by ρ_{i+1}^k. The number ρ_k is called a k-dimensional *Betti number*, and the numbers ρ_i^k k-dimensional *torsion coefficients of the space* X.

In spite of a somewhat complex character of its substantiation, the described method is quite convenient from the practical point of view. It enables us to compute homology groups of a good deal of concrete spaces simply.

Exercise 1°. Representing the sphere S^n as a cell complex $S^n = e^n \cup e^0$, $n \geqslant 1$, compute the homology groups of S^n. Show that $\rho_k = 0$, $k \neq 0, n$; $\rho_0 = \rho_n = 1$ and that each $\rho_i^k = 0$.

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* Just like in Sec. 4, we assume here that G is a ring with identity.

The homology groups of the complex projective space CP^n will be calculated in the following way. At first, we define a smooth function f on the manifold CP^n whose all critical points are nondegenerate, and then establish with its help the structure of the cell complex which is homotopy equivalent to CP^n while calculating its homology groups.

We will consider CP^n as the orbit space of the group S^1 acting on S^{2n+1}. We define the function $\varphi : C^{n+1} \to R^1$ by putting $\varphi(z_0, \ldots, z_n) = \sum\limits_{j=0}^{n} c_j |z_j|^2$, where c_j are real numbers with $c_j < c_{j+1}$. Let $(z_0, \ldots, z_n) \in S^{2n+1} \subset C^{n+1}$, i.e., $\sum\limits_{j=0}^{n} |z_j|^2 = 1$. It is easy to see that for any complex number λ such that $|\lambda| = 1$, the equality $\varphi(z_0, \ldots, z_n) = \varphi(\lambda z_0, \ldots, \lambda z_n)$ holds. Thus, φ defines a function on CP^n. Denote it by $f : CP^n \to R^1$.

We now construct on CP^n the following local coordinate system. Let U_j be the set of equivalence classes of points $(z_0, \ldots, z_n) \in S^{2n+1}$ such that $z_j \neq 0$. Put $|z_j| \cdot \dfrac{z_k}{z_j} = x_{jk} + iy_{jk}$. The functions

$$x_{jk}(z_0, \ldots, z_n), \qquad y_{jk}(z_0, \ldots, z_n), \qquad k = 0, \ldots, j-1, j+1, \ldots, n,$$

define a diffeomorphism of the set U_j onto the unit ball in R^{2n}.

Exercise 2°. Verify that the sets U_j and mappings given by the functions

$$x_{jk}, y_{jk}, k = 0, \ldots, j-1, j+1, \ldots, n, \qquad j = 0, 1, \ldots, n,$$

form an atlas on the smooth manifold CP^n.

Since $|z_k|^2 = x_{jk}^2 + y_{jk}^2$ and $|z_j|^2 = 1 - \sum\limits_{k \neq j} (x_{jk}^2 + y_{jk}^2)$, the function f can be represented in local coordinates in U_j in the form

$$f(\ldots, x_{jk}, y_{jk}, \ldots) = c_j + \sum_{k \neq j} (c_k - c_j)(x_{jk}^2 + y_{jk}^2).$$

The only critical point of the function f in U_j is the point $x_{jk} = y_{jk} = 0$, $k = 0, 1, \ldots, j-1, j+1, \ldots, n$. This critical point is nondegenerate, and its index equals twice the number of negative differences $c_k - c_j$, i.e., twice the number of those numbers c_k which are less than c_j. Therefore, the index of a critical point in U_0 equals zero, that of a critical point in U_1 equals two, etc. In general, the index of a critical point in U_j is $2j$.

Thus, the function f has n critical points whose indices equal $2j$, $0 \leqslant j \leqslant n$. Therefore, the space CP^n has the homotopy type of the cell complex K consisting of cells of even dimensions $2j$, $0 \leqslant j \leqslant n$, one of each dimension (see Sec. 12, Ch. IV). For such a complex K, we have

$$\bar{C}_k(K; G) = \begin{cases} G & \text{when } k = 2j \leqslant 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Since one of the groups $\check{C}_k(K; G)$, $\check{C}_{k-1}(K; G)$ is trivial, in the complex $\check{C}_*(K; G)$ consisting of the groups $\check{C}_k(K; G)$, the differential can only be trivial. We obtain the isomorphism

$$H_k^s(K; G) \approx \check{C}_k(K; G).$$

Taking into account the fact that the homology groups of spaces which are homotopy equivalent coincide, we have the final result:

$$H_k^s(CP^n; G) = \begin{cases} G & \text{when} \quad k = 2j \leqslant 2n \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 3°. Show that if M^n is a compact, smooth manifold of dimension n, then $H_k^s(M^n; G) = 0$ when $k > n$.

8. EULER CHARACTERISTIC AND LEFSCHETZ NUMBER

Quite important in applications is the question when a continuous mapping $f: X \to X$ of a topological space X into itself has a fixed point, i.e., when a point $x \in X$ such that $f(x) = x$ exists. Sufficient conditions for the existence of fixed points can be given in terms of homology groups and their homomorphisms. The present section is devoted to these questions. Throughout below, we will consider topological spaces which are compact polyhedra.

1. The Lefschetz Number of a Simplicial Mapping. Henceforward, we will assume the coefficient group G to be a field. Consider a simplicial mapping $f: |K| \to |K|$, where in accordance with the agreement of Sec. 3, K is a finite simplicial complex. The induced homomorphism

$$f_{*p}: H_p(K; G) \to H_p(K; G)$$

is an endomorphism of the vector space $H_p(K; G)$. The choice of a basis in $H_p(K; G)$ enables us to associate the endomorphism with a matrix whose spur $\mathrm{Sp}\,(f_{*p})$ does not depend on the choice of a basis.

DEFINITION 1. *The Lefschetz number of a simplicial mapping* $f: |K| \to |K|$ of a compact polyhedron $|K|$ into itself is the quantity

$$\Lambda_f = \sum_{p=0}^{\infty} (-1)^p \,\mathrm{Sp}\,(f_{*p}). \tag{1}$$

Due to the finiteness of the simplicial complex K, sum (1) is the sum of a finite number of addends (the finiteness of K is also required for $H_p(K; G)$ to be finite-dimensional vector spaces and for the definitions of the spurs $\mathrm{Sp}\,(f_{*p})$ to be correct).

Denote the spur of the endomorphism matrix

$$\hat{f}_p: C_p(K; G) \to C_p(K; G)$$

of the vector space $C_p(K; G)$ by $\mathrm{Sp}\,(\hat{f}_p)$.

THEOREM 1. *For a simplicial mapping f, the formula is valid*

$$\Lambda_f = \sum_{p=0}^{\infty} (-1)^p \operatorname{Sp}(\hat{f}_p). \tag{2}$$

Theorem 1 states that the alternating sum of the spurs of a chain complex endomorphism equals the alternating sum of the spurs of the induced homology group endomorphism.

To prove Theorem 1, the following two lemmata are necessary which are an easy exercise from linear algebra.

LEMMA 1. *Let $A : E \to E$ be an endomorphism of a vector space E, E_0 a vector subspace of the space E and $AE_0 \subset E_0$. Then A defines the endomorphism $\tilde{A} : E/E_0 \to E/E_0$ and*

$$\operatorname{Sp}(A) = \operatorname{Sp}(A|_{E_0}) + \operatorname{Sp}(\tilde{A}). \tag{3}$$

LEMMA 2. *Let $\Delta : E \to F$ be an isomorphism of vector spaces and $A : E \to E$, $B : F \to F$ operators such that the diagram*

$$\begin{tikzcd} E \arrow[r, "\Delta"] \arrow[d, "A"'] & F \arrow[d, "B"] \\ E \arrow[r, "\Delta"] & F \end{tikzcd}$$

is commutative, i.e., $B\Delta = \Delta A$; then

$$\operatorname{Sp}(A) = \operatorname{Sp}(B). \tag{4}$$

THE PROOF OF THEOREM 1. Since $\hat{f}_* : C_*(K; G) \to C_*(K; G)$ is a chain complex homomorphism, $\hat{f}_p(\operatorname{Ker} \partial_p) \subset \operatorname{Ker} \partial_p$ and $\hat{f}_p(\operatorname{Im} \partial_{p+1}) \subset \operatorname{Im} \partial_{p+1}$.

Let us introduce the following designations:

$$\operatorname{Ker} \partial_p = Z_p, \quad \operatorname{Im} \partial_{p+1} = B_p, \quad C_p(K; G)/\operatorname{Ker} \partial_p = T_p, \quad Z_p/B_p = H_p.$$

By Lemma 1, we have

$$\operatorname{Sp}(\hat{f}_p) = \operatorname{Sp}(\hat{f}_p|_{Z_p}) + \operatorname{Sp}(\tilde{\hat{f}}_p|_{T_p}) = \operatorname{Sp}(\hat{f}_p|_{B_p}) + \operatorname{Sp}(\tilde{\hat{f}}_p|_{H_p}) + \operatorname{Sp}(\overset{\approx}{\hat{f}}_p|_{T_p}) \tag{5}$$

But the differential ∂_p induces a canonical isomorphism $\tilde{\partial}_p : T_p \to B_{p-1}$, and moreover, the diagram

$$\begin{tikzcd} T_p \arrow[r, "\partial_p"] \arrow[d, "\overset{\approx}{f}_p"'] & B_{p-1} \arrow[d, "\hat{f}_{p-1}"] \\ T_p \arrow[r, "\partial_p"] & B_{p-1} \end{tikzcd}$$

is commutative. By Lemma 2, we obtain

$$\operatorname{Sp}(\overset{\approx}{\hat{f}}_p|_{T_p}) = \operatorname{Sp}(\hat{f}_{p-1}|_{B_{p-1}}). \tag{6}$$

Because $C_0(K; G) = \operatorname{Ker} \partial_0$, we have

$$\operatorname{Sp} (\tilde{\hat{f}}_0|_{T_0}) = \operatorname{Sp} (\tilde{\hat{f}}|_0) = 0. \tag{7}$$

It is clear that by definition the homomorphism $\tilde{\hat{f}}: H_p \to H_p$ coincides with the homomorphism $f_{*p}: H_p(K; G) \to H_p(K; G)$. Thus, from equalities (5), (6), (7), we have

$$\operatorname{Sp} (\hat{f}_p) = \operatorname{Sp} (\hat{f}_p|_{B_p}) + \operatorname{Sp} (f_{*p}) + \operatorname{Sp} (\hat{f}_{p-1}|_{B_{p-1}}),$$

whence

$$\sum_{p=0}^{\infty} (-1)^p \operatorname{Sp} (\hat{f}_p) = \sum_{p=0}^{\infty} (-1)^p [\operatorname{Sp} (\hat{f}_p|_{B_p}) + \operatorname{Sp} (f_{*p}) + \operatorname{Sp} (\hat{f}_{p-1}|_{B_{p-1}})] = \sum_{p=0}^{\infty} (-1)^p \operatorname{Sp} (f_{*p});$$

thus,

$$\sum_{p=0}^{\infty} (-1)^p \operatorname{Sp} (\hat{f}_p) = \sum_{p=0}^{\infty} (-1)^p \operatorname{Sp} (f_{*p}). \blacksquare \tag{8}$$

COROLLARY. *The Lefschetz number Λ_f over the coefficient field of characteristic zero and, in particular, over the fields Q, R or C, is an integer.*

In fact, having considered in $C_p(K; G)$ a basis consisting of oriented simplexes, we obtain that $\operatorname{Sp} (\hat{f}_p)$ is an integer, and therefore, due to (4), Λ_f is an integer. ■

The role of the number Λ_f is disclosed by the following theorem.

THEOREM 2. *Let $f: |K| \to |K|$ be a simplicial mapping and $\Lambda_f \neq 0$. Then there exists a fixed point of the mapping f, i.e., a point $x \in K$ such that $f(x) = x$.*

PROOF. Due to (2), it follows from the condition $\Lambda_f \neq 0$ that $\sum_{p=0}^{\infty} (-1)^p \operatorname{Sp} (\hat{f}_p) \neq 0$, therefore there is p such that $\operatorname{Sp} (\hat{f}_p) \neq 0$. With respect to a basis consisting of oriented simplexes, the matrix of the endomorphism $\hat{f}_p$ consists of elements equal to 0, +1 and −1. Since $\operatorname{Sp} (\hat{f}_p) \neq 0$, there is $\tau_i^p \in K$ such that $\hat{f}_p[\tau_i^p] = \pm[\tau_i^p]$. Therefore $f|_{\tau_i^p}$ is a homeomorphism of τ_i^p onto itself which is linear in barycentric coordinates, whence the barycentre τ_i^p is a fixed point of the mapping f. ■

Let us discuss an important complement to Theorem 2. Let $f: |K^{(r)}| \to |K|$ be a simplicial mapping: it is certainly a continuous mapping $f: |K| \to |K|$, but not necessarily simplicial. We introduce the following superposition $\Theta_*^{(r)} \hat{f}_*$ (see Sec. 3) of chain homomorphisms

$$C_*(K^{(r)}; G) \xrightarrow{\hat{f}_*} C_*(K; G) \xrightarrow{\Theta_*^{(r)}} C_*(K^{(r)}; G).$$

The chain homomorphism $\Theta_*^{(r)} \hat{f}_*$ induces the homology group homomorphism

$$(\Theta_*^{(r)}\hat{f}_*)_* : H_*(K^{(r)}; G) \to H_*(K^{(r)}; G).$$

By definition, we put

$$\Lambda_f = \sum_{p=0}^{\infty} (-1)^p \operatorname{Sp} [\Theta_*^{(r)}\hat{f}_*)_{*p}]. \tag{9}$$

Exercise 1°. Prove that

$$\sum_{p=0}^{\infty} (-1)^p \operatorname{Sp} (\Theta_p^{(r)}\hat{f}_p) = \sum_{p=0}^{\infty} (-1)^p \operatorname{Sp} [\Theta_*^{(r)}\hat{f}_*)_{*p}] \tag{10}$$

and that if $\Lambda_f \neq 0$, then there exist simplexes $\tau^p \in K^{(r)}$ and $\mu^p \in K$, such that $\tau^p \subset \mu^p$ and $f(\tau^p) = \mu^p$.

Now, we consider an example when $f = 1_K : |K| \to |K|$ is the identity mapping of the polyhedron $|K|$. Denote the dimension of the vector space $H_p(|K|; G)$ * by β^p, and the number of p-dimensional simplexes in the simplic. , complex K by d_p. It is obvious that

$$\operatorname{Sp} ((1_K))_{*p} = \beta^p, \quad \operatorname{Sp} ((\hat{1}_K)_p) = \operatorname{Sp} (1_{C_p(K;G)}) = d_p.$$

Formula (8) becomes as follows

$$\sum_{p=0}^{\infty} (-1)^p d_p = \sum_{p=0}^{\infty} (-1)^p \beta^p. \tag{11}$$

Formula (11) establishes a relation between the geometric and homological characteristics of a polyhedron.

DEFINITION 2. The *Euler characteristic* of a compact polyhedron $|K|$ is the quantity

$$\chi(|K|) \overset{\text{def}}{=} \sum_{p=0}^{\infty} (-1)^p d_p = \sum_{p=0}^{\infty} (-1)^p \beta^p. \tag{12}$$

It is clear that $\chi(|K|) = \Lambda_{1|K|}$.

Exercise 2°. Show that the equality $\chi(S^n) = 1 + (-1)^n$ is valid.

2. The Lefschetz Number of a Continuous Mapping.

In the previous reasoning, we considered only simplicial mappings. But the construction of a Lefschetz number and the statement of Theorem 2 can be generalized also for arbitrary continuous mappings. In doing so, we will use the uniqueness theorem of homology theory (see Sec. 5) and the method of approximation of a continuous mapping of a polyhedron by a simplicial mapping.

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* If G is a field of characteristic zero then β^p coincides with the p-dimensional Betti number ρ_p of the space $|K|$ in the sense of the definition in Sec. 7.

THEOREM 3 (THE SIMPLICIAL APPROXIMATION THEOREM) *Let $X = |L|$ be a compact polyhedron and $f: X \to X$ a continuous mapping. Then for any $\varepsilon > 0$, there are a triangulation K of the polyhedron X, its r-th barycentric subdivision $K^{(r)}$ and a simplicial mapping $f_\varepsilon: |K^{(r)}| \to |K|$ such that for any point $x \in X$, the inequality $\rho(f(x), f_\varepsilon(x)) < \varepsilon$ holds.*

PROOF We select on the polyhedron X a triangulation K such that the fineness of the triangulation K is less than ε. We will call the interior of the union of all simplexes, whose vertices are $a \in K$, the star St a with the vertex a. It is obvious that the stars of all vertices of K form a covering of X; the inverse images $\{f^{-1}(\mathrm{St}\, a^p)\}_{a^p \in K}$ also form a covering.

Since X is compact, by the Lebesgue number lemma (see Theorem 13, Sec. 13, Ch. II), there exists a number $\nu > 0$ such that any set of diameter $\delta < \nu$ is contained in one of the sets $f^{-1}(\mathrm{St}\, a^p)$. We select r such that the fineness of $K^{(r)}$ is less than $\nu/2$. Then the mapping f transforms any star St b^q, $b^q \in K^{(r)}$ into a certain star St a^p, $a^p \in K$. We define a simplicial mapping $f_\varepsilon: |K^{(r)}| \to |K|$ by the equalities

$$f_\varepsilon(b^q) = a^p. \tag{13}$$

Exersice 3°. Verify that formula (13) does define a simplicial mapping.

Now, we calculate $\rho(f(x), f_\varepsilon(x))$ for $x \in X$. If x is a vertex of $K^{(r)}$ then $f(\mathrm{St}\, x) \subset \mathrm{St}\, a^p$, $a^p \in K$, and, in particular, $f(x) \in \mathrm{St}\, a^p$, therefore

$$\rho(f(x), a^p) = \rho(f(x), f_\varepsilon(x)) < \varepsilon.$$

If, however, $x \in \mathrm{Int}\,(b^0, \ldots, b^q)$, where $(b^0, \ldots, b^q) \in K^{(r)}$, then $x \in \bigcap_{i=0}^{q} \mathrm{St}\, b^i$. We have $f(x) \in \bigcap_{a^i = f_\varepsilon(b^i)} \mathrm{St}\, a^i$, hence $f(x)$ lies in the simplex determined by the vertices $a^i = f_\varepsilon(b^i)$. Since f_ε is a simplicial mapping, $f_\varepsilon(x)$ gets into the same simplex from K. Thus, in this case also $\rho(f(x), f_\varepsilon(x)) < \varepsilon$. ■

Exercises.

4°. Show that the mapping f_ε is homotopic to the mapping f.

5°. Show that for a compact polyhedron X, there exists a positive number $\alpha = \alpha(X)$ such that from the inequality $\rho(f(x), g(x)) < \alpha$ satisfied for each $x \in X$ (f, $g: X \to X$ being continuous mappings), it follows that the mappings f and g are homotopic.

Due to the uniqueness theorem of homology theory, the isomorphism $H_*(K; G) = H^s_*(X; G)$ holds for the compact polyhedron $X = |K|$, and therefore $\dim_G \bigoplus_p H^s_p(X; G) < \infty$. Consequently, the following definition may be given.

DEFINITION 3. *The Lefschetz number of a continuous mapping $f: X \to X$* of a compact polyhedron X into itself is

$$\Lambda_f = \sum_{p=0}^{\infty} (-1)^p \,\mathrm{Sp}\,(f^s_{*p}),$$

where

$$f^s_{*p} : H^s_p(X; G) \to H^s_p(X; G). \tag{14}$$

Due to the uniqueness theorem of homology theory, for a simplicial mapping $f: |K^{(r)}| \to |K|$, $r \geqslant 1$, the equality

$$\sum_{p=0}^{\infty} (-1)^p \operatorname{Sp}(f^s_{*p}) = \sum_{p=0}^{\infty} (-1)^p \operatorname{Sp}[(\theta^r_* \hat{f}_*)_{*p}] \tag{15}$$

holds. Thus, Definitions 1 and 3 are consistent.

It is obvious that for homotopic continuous mappings $f, g: X \to X$, we have $\Lambda_f = \Lambda_g$. Therefore the Lefschetz number of the mapping $f: X \to X$ equals that of its simplicial approximation $f_\varepsilon: |K^{(r)}| \to |K|$, where K is a triangulation of X. The Lefschetz number of continuous mapping could be defined as that of its simplicial approximation without the use of singular homology theory.

The following theorem is quite useful for various applications. In its proof, we shall use the uniqueness theorem of homology theory.

THEOREM 4 (LEFSCHETZ). *Let $f: X \to X$ be a continuous mapping of a compact polyhedron $X = |L|$ into itself, and $\Lambda_f \neq 0$. Then there exists a fixed point of the mapping f, i.e. a point $x \in X$ such that $f(x) = x$.*

PROOF. Assume that f has no fixed points. Then there is $\beta > 0$ such that $\rho(f(x), x) \geqslant \beta$ for each $x \in X$. Let $\gamma = \min(\beta, \alpha(X))$ (see Ex. 5).

Consider a triangulation K of fineness $\gamma/3$ and a simplicial $\gamma/3$-approximation $f_{\gamma/3}$ of the mapping f. For arbitrary points x, y of any simplex $\tau^q \in K^{(r)}$, we have the inequalities

$$\rho(f_{\gamma/3}(x), y) \geqslant \rho(f(x), x) - \rho(x, y) - \rho(f_{\gamma/3}(x), f(x)) \geqslant \gamma/3.$$

This means that the relation $\tau^q \subset f_{\gamma/3}(\tau^q)$ is impossible. On the other hand, due to $\Lambda_{f\gamma/3} = \Lambda_f \neq 0$, there is $\tau^q \in K^{(r)}$ for which such a relation is valid (see Exercises 1 and 5).

The obtained contradiction completes the proof of the theorem. ∎

Exercises

6°. Extend Definition 2 and Theorem 4 to a compact polyhedron X homeomorphic to $|L|$.

7°. Verify that $\Lambda_f = 1$ with the conditions of the Brouwer fixed-point theorem (see Sec. 4, Ch. III).

3. The Euler Characteristic of a Manifold and Singular Points of a Vector Field.

We now dwell on the application of the obtained results to manifold theory.

THEOREM 5. *Let M^n be both a smooth, compact manifold and a polyhedron *. Let $\chi(M^n) \neq 0$. Then for any vector field X on M^n, there exists a point $x^0 \in M^n$ such that $X(x_0) = 0$.*

In other words, there is no vector field without zeroes on a manifold with a nonzero Euler characteristic.

* Note that all smooth, compact manifolds are polyhedra.

PROOF. As it was mentioned in Sec. 8, Ch. IV, for a vector field X, there exists a one-parameter family of diffeomorphisms $U(x, t)$ such that $U(x, 0) \equiv x$, and the field X is its infinitesimal generator. Moreover, the orbit $\bigcup_t U(x, t)$ of the point x is called the integral curve of the field X at the point x. It is easy to see that the family of diffeomorphisms $U(x, t)$, $0 \leqslant t \leqslant t_0$, carries out the homotopy between the diffeomorphisms $U_0 = 1_{M^n}$ and $U_{t_0}: M^n \to M^n$, where $U_{t_0}(x) \overset{\text{def}}{=} U(x, t_0)$. Therefore, $\Lambda_{1_{M^n}} = \Lambda_{U_{t_0}}$, but $\Lambda_{1_{M^n}} = \chi(M^n)$; thus, for any t_0, we obtain $\Lambda_{U_{t_0}} = \chi(M^n) \neq 0$. Therefore, the diffeomorphism U_{t_0} possesses a fixed point (for each t_0) (see Theorem 4).

Assume now that the field X vanishes nowhere on M^n. Then, since M^n is compact, for a certain $\beta > 0$, a sufficiently small $\alpha > 0$, any $x \in M^n$, and in the Riemannian metric, the inequality $\beta \geqslant \langle X(x), X(x) \rangle \geqslant \alpha$ holds. Hence, any point $x \in M^n$ is unfailingly shifted by the diffeomorphism U_t along the integral curve of the point x for a sufficiently small $t > 0$; this can be checked by considering the integral curve in the chart at the point x. The last statement is contrary to the existence of a fixed point for the diffeomorphism U_t. ■

COROLLARY. *If n is even then there is not a single vector field without zeroes on the sphere S^n.*

LEMMA 3. *There exists a smooth vector field on a compact smooth manifold and the sum of the indices of singular points of this field equals the Euler characteristic of the manifold.*

PROOF. Let M^n be a compact, smooth manifold, $f: M^n \to R^1$ a Morse function (a smooth function whose all critical points are nondegenerate). The space M^n has the homotopy type of a cell complex K, the number of cells of dimension λ of which equals the number $m(\lambda)$ of critical points x_i^λ of index λ of the function f (see Sec. 11, Ch. IV). The Euler characteristic $\chi(K)$ of the space K equals

$$\sum_{\lambda=0}^{\infty} (-1)^\lambda \dim_G \check{C}_\lambda(K; G) = \sum_{\gamma=0}^{\infty} (-1)^\lambda \dim_G H_\lambda^s(K; G)$$

(compare with Definition 2 and Theorem 1 of the present section). Thus,

$$\chi(M^n) = \chi(K) = \sum_{\lambda=0}^{\infty} (-1)^\lambda m(\lambda). \tag{16}$$

On the other hand, due to Exercise 10, Sec. 6, the index of the singular point x_i^λ of the gradient field equals $(-1)^\lambda$. Therefore, $\sum_{\lambda=0}^{\infty} (-1)^\lambda m(\lambda)$ is the sum of the indices of singular points of the gradient field of the function f. ■

LEMMA 4. *The sum of the indices of singular points of a vector field with isolated*

singular points on a compact, smooth manifold does not depend on the choice of the vector field.

We give the proof of this lemma in a nutshell. Let M^n be a connected manifold embedded in R^m, $m > n + 1$. We select a sufficiently small 'tubular' neighbourhood of the manifold M^n in R^m, i.e., a neighbourhood $U(M^n)$ which is the total space of a locally trivial fibre space with the base space M^n and a fibre homeomorphic to the disc D^{m-n}. Moreover, the projection r of this fibre bundle is a smooth retraction, and the manifold M^n is a strong deformation retract of the space $U(M^n)$. Intuitively, the tubular neighbourhood of the manifold M^n can be imagined to consist of discs $D_\varepsilon^{m-n}(x)$ over each point $x \in M^n$ that lie in $(m-n)$-dimensional planes orthogonal to the tangent planes of the manifold M^n. The set $\overline{U(M^n)}$ is a compact polyhedron. It is not complicated to show that $H_{m-1}^s(\partial U(M^n); Z) \simeq Z$. The generator of this group is a cycle bounding $U(M^n)$. Therefore any mapping $\varphi : \partial U(M^n) \to S^{m-1}$ determines an element $\deg \varphi \in Z$. Consider a certain field $\Phi : \overline{U(M^n)} \to R^m$ which does not vanish on $\partial U(M^n)$. We associate the field Φ with the normed mapping

$$\bar{\Phi} : \partial U(M^n) \to S^{m-1}, \qquad \bar{\Phi} x = \Phi x / \|\Phi x\|.$$

The degree $\deg \bar{\Phi}$ of the mapping $\bar{\Phi}$ equals the sum of the indices of singular points of the field Φ. Now, let v be a vector field on the manifold M^n. We define the field $w : U(M^n) \to R^m$ by the formula $w(x) = v(rx) + x - r(x)$. The sum of the indices of singular points of the field w coincides with the sum of the indices of singular points of the tangent field v (by means of the Sard theorem, the general case may be reduced to the study of smooth fields with nondegenerate singular points, and the application of the result of Exercise 7, Sec. 6). The field w on $\partial U(M^n)$ is homotopic, without singular points, to the vector field $z(x) = x - r(x)$. Hence, for the normed mappings $\bar{w}, \bar{z}$, we obtain $\deg \bar{w} = \deg \bar{z}$ and therefore $\deg \bar{w}$ does not depend on the field v.

Lemmata 3 and 4 lead to the following theorem.

THEOREM 6. *The sum of the indices of singular points of a vector field with isolated singular points on a compact, smooth manifold equals the Euler characteristic of the manifold.*

Exercise 8°. Let M^n be a compact, smooth manifold, and $\beta^p(M^n) \overset{\text{def}}{=} \dim_G H_p^s(M^n; G) \neq 0$. Show that any Morse function on the manifold M^n has not less than $\beta^p(M^n)$ critical points of index p (Morse inequalities).

FURTHER READING

In the last decade, there appeared several monographs providing a systematic approach to homology theory and its applications. We indicate, first of all, *Algebraic Topology* [73] by Spanier *Lectures on Algebraic Topology* [27] by Dold, *Homology and Cohomology. Theory* [53] by Massey as those most corresponding to the demands of today. Recommending them for a profound and systematic study

of homology theory, we emphasize, however, that intrinsically they are rather auxiliary textbooks concentrated on special courses.

Sec. 1. While studying thoroughly separate topics touched upon in the present chapter, it will be undoubtedly interesting for the reader to turn his attention to the following literature: the notion of homology was introduced and elaborated in the classical *Analysis sitûs* and the five complements to it by Poincaré (see the sixth volume of *Oeuvres de Henri Poincaré* [63]).

Sec. 2. To study chain complexes and their homology groups, the reader is advised to see Ch. II of *Homology* [51] by MacLane.

Sec. 3. Simplicial homology theory is compactly and thoroughly expounded in *Outline of Combinatorial Topology* [65] by Pontryagin. Quite useful is also the acquaintance with *Introduction to Homological Dimension Theory and Combinatorial Topology* [2] (Chs. I-II) by Alexandrov and *Homology Theory* [40] (Chs. I-III) by Hilton and Wylie.

Sec. 4. A brief and geometry-oriented presentation of singular homology theory is given in *Homotopy Theory* [33] (Ch. II) by Fuchs et al. The elements of singular theory are given compactly and with sufficient rigour in *Elemente de topologie şi varietăi difereniabile* [79] (Ch. II) by Teleman. The presentation of singular theory is thorough in the above-mentioned book [40] (Ch. VIII). Considerable attention is given there to the technical details of the theory. In the proof of the theorem on homomorphisms induced by homotopy mappings, we followed MacLane [51] (Ch. II, Sec. 8) and Massey [53], since this method enables us not to introduce certain concepts generally used in extensive courses. The reader can study the relation between homology and homotopy groups in the above-mentioned books by Teleman [79] (Ch. IV, Secs. 3, 7), Hilton and Wylie [40] (Sec. 8.8) and Fuchs et al. [33] (Sec. 13), and also in *Homotopy Theory* [43] (Ch. II, Sec. 6 and Ch. V, Sec. 4) by Hu S.-T.

Sec. 5. The axiomatic approach to homology theory is given in *Foundations of Algebraic Topology* [30] by Eilenberg and Steenrod. A direct proof of the equivalence of simplicial and singular theories on the category of polyhedra is given, e.g., in the book by Hilton and Wylie [40] (Sec. 8.6). The reader may find Alexandrov-Čech homology theory in the book by Teleman [79] (Ch. II, Sec. 18) and, in greater detail, in the books by Alexandrov [2], [3].

Sec. 6. The homology groups of spheres are calculated in all the courses of homology theory. We followed the book by Fuchs et al. [33] (Secs. 12-13). In the above-mentioned *Combinatorial Topology* [1] (Ch. XVI) by Alexandrov, the theories of the degree of a mapping, characteristic of a vector field and index of a singular point are given extensively and on the basis of simplicial homology theory.

Sec. 7. As regards cell homology theory, we recommend the lectures by Boltyansky *on Basic Concepts of Algebraic Topology* [15] in which a thorough elaboration of cell theory is given, and also the books by Fuchs et al. [33] (Sec. 12) and Teleman [79] (Ch. VI).

Sec. 8. The notions of the Euler characteristic and Lefschetz number are systematically expounded in the book by Pontryagin [65] (Secs. 6 and 16) and also in the books by Alexandrov [1] (Ch. XVII) and Hilton and Wylie [40] (Sec. 5.8). The proof of the theorem on a triangulation of smooth manifolds can be found in

Geometric Integration Theory by Whitney (Ch. IV) [82] and in *Elementary Differential Topology* [60] by Muncres. The reader may find the theory regarding the sum of the indices of a vector field singular points on a manifold in *Morse Theory* [54] (Sec. 6) and *Topology from the Differential Viewpoint* [55] (Sec. 6) by Milnor Topics related to singular points of vector fields on a manifold are very important for the theory of differential equations on a manifold. An introduction to this theory is given, e.g., in *Ordinary Differential Equations* [10] (Ch. V) by Arnold.

As a book of problems in homology theory, the work by Novikov et al. *Problems in Geometry* [61] may be taken.

ILLUSTRATION TO CHAPTER I

The central part of the picture illustrates the standard embedding chain of crystalline groups of the three-dimensional Euclidean space: their standard groups, embedded into each other are depicted as the fundamental domains (platonic bodies: a cube, a tetrahedron, a dodecahedron). The platonic bodies are depicted classically, i.e., their canonical form is given; they are supported by two-dimensional surfaces (leaves), among which we discern the projective plane (cross-cap), and spheres with handles. The fantastic shapes and interlacings (as compared with the canonical objects) symbolize the topological equivalence.

At the top, branch points of Riemann surfaces of various multiplicities are depicted: on the right, those of the Riemann surfaces of the functions $w = \sqrt[5]{z}$ and $w = \sqrt{z}$; on the left below, that of the same function $w = \sqrt{z}$, and over it, a manifold with boundary realizing a bordism mod 3.

ILLUSTRATION TO CHAPTER II

The figure occupying most of the picture illustrates the construction of a topological space widely used in topology, i.e., a 2-adic solenoid possessing many interesting extremal properties. The following figures are depicted there: the first solid torus is shaded, the second is white, the third is shaded in dotted lines and the fourth is shaded doubly. To obtain the 2-adic solenoid, it is necessary to take an infinite sequence of nested solid tori, each of which encompasses previous twice along its parallel, and to form their intersection.

Inside the opening, a torus and a sphere with two handles are shown. The artist's skill and his profound knowledge of geometry made it possible to represent complex interlacing of the four nested solid tori accurately.

ILLUSTRATION TO CHAPTER III

The canonical embedding of a surface of genus g into the three-dimensional Euclid in space is represented on the right. A homeomorphic embedding of the same surface is shown on the left. The two objects are homeomorphic, homotopic and even isotopic. The artist is a mathematician and he has chosen these two, very much different in their appearance, from an infinite set of homeomorphic images.

ILLUSTRATION TO CHAPTER IV

Here an infinite total space of a covering over a two-dimensional surface, viz., a sphere with two handles, is depicted. The artist imparted the figure the shape of a python and made the base space of the covering look very intricate. Packing spheres into the three-dimensional Euclidean space and a figure homeomorphic to the torus are depicted outside the central object. The mathematical objects are placed so as to create a fantastic landscape.

ILLUSTRATION TO CHAPTER V

A regular immersion of the projective plane RP^2 in R^3 is represented in the centre on the black background. The largest figure is the Klein bottle (studied in topology as a non-orientable surface) cut in two (Möbius strips) along a generator

by a plane depicted farther right along with the line of intersection; the lower part is plunging downwards; the upper part is being deformed (by lifting the curve of intersection and building the surface up) into a surface with boundary S^1; a disc is being glued to the last, which yields the surface RP^2. The indicated immersion process can be also used for turning S^2 'inside out' into R^3.

On the outskirts of the picture, a triangulation of a part of the Klein bottle surface is represented.

A detailed explanation of this picture may serve as a material for as much as a lecture in visual topology.

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